

# Twisted Algebras of Multiplier Hopf $(*)$ -algebras

Shuanhong Wang

Department of Mathematics, Southeast University

Nanjing, Jiangsu 210096, P. R. of China

E-mail: shuanhwang@seu.edu.cn

## ABSTRACT

In this paper we study twisted algebras of multiplier Hopf  $(*)$ -algebras which generalize all kinds of smash products such as generalized smash products, twisted smash products, diagonal crossed products, L-R-smash products, two-sided crossed products and two-sided smash products for the ordinary Hopf algebras appeared in [P-O].

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**Key words:** Twisted tensor product; Twisted algebra; Multiplier Hopf  $(*)$ -algebra.

## 1. Introduction

In [P-O], Panaite and Van Oystaeyen introduced a more general version of the so-called L-R-smash product for (quasi-)Hopf algebras and studied its relations with other kinds of crossed products (two-sided smash and crossed product and diagonal crossed product), In order to find the method of constructing multiplier Hopf  $(*)$ -algebras, in this paper we will try to generalize these results to the situation for multiplier Hopf algebras introduced in [VD1-VD3].

This paper is organized as follows. In Section 2 of this paper we will study the notions of a generalized twisted tensor product, a generalized twisted smash product and a generalized L-R-smash product. We also study some isomorphisms between

them. In Section 3 we focus on two-times twisted tensor products for multiplier Hopf algebras. Finally, in Section 4 we will introduce the notion of a Long module algebra and study the the twisted product of the given multiplier Hopf algebras.

## 1. Preliminaries

Let  $A$  be an associative algebra with a nondegenerate product. If  $A$  has a unit element, this requirement is automatically satisfied. In the general case, we only suppose the existence of local units in the following sense. Let  $\{e_1, \dots, e_n\}$  be a finite set of elements in  $A$ . Then there exist elements  $e, f \in A$  so that  $ee_i = e_i = e_if$  for  $i = 1, \dots, n$ .

A multiplier  $m = (m_1, m_2)$  of the algebra  $A$  is a pair of linear mappings in  $End_k(A)$  such that  $m_2(a)b = am_1(b)$  for all  $a, b \in A$ . The set of multipliers of  $A$  is denoted by  $M(A)$ . It is a unital algebra which contains  $A$  as essential ideal through the embedding  $a \mapsto (a \cdot, \cdot a)$ . Therefore  $m \cdot a = (m_1(a) \cdot, \cdot m_1(a)) \equiv m_1(a)$  and  $a \cdot m = (m_2(a) \cdot, \cdot m_2(a)) \equiv m_2(a)$  for all  $m \in M(A)$  and  $a \in A$ . Hence we will frequently use the identification  $a \cdot m = m_2(a)$  and  $m \cdot a = m_1(a)$ . If  $A$  is unital then  $A = M(A)$ . If  $A$  is a  $*$ -algebra then  $M(A)$  is a  $*$ -algebra through  $m^* = (m_2^*, m_1^*)$  where  $\psi^*(a) := \psi(a^*)^*$  for any  $a \in A$ ,  $\psi \in End_k(A)$ . Since the multiplication of  $A$  is supposed to be nondegenerate a multiplier  $m = (m_1, m_2)$  of  $A$  is uniquely determined by its first or second component. For a tensor product of two algebras  $A$  and  $B$  one obtains the canonical algebra embeddings  $A \otimes B \hookrightarrow M(A) \otimes M(B) \hookrightarrow M(A \otimes B)$ .

Let  $A$  be an algebra with a non-degenerate product. An algebra homomorphism  $\Delta : A \longrightarrow M(A \otimes A)$  is called a comultiplication on  $A$  if

- (i)  $T_1(x \otimes y) = \Delta(x)(1 \otimes y) \in A \otimes A$  and  $T_2(x \otimes y) = (x \otimes 1)\Delta(y) \in A \otimes A$  for all  $x, y \in A$  (some times we call this condition a "coving theory");
- (ii)  $(T_2 \otimes id)(id \otimes T_1) = (id \otimes T_1)(T_2 \otimes id)$  on  $A \otimes A \otimes A$ .

We call  $A$  with a comultiplication  $\Delta$  a multiplier Hopf algebra if the linear maps  $T_1, T_2$  on  $A \otimes A$  are bijective. Moreover, we call  $A$  regular if  $\tau\Delta$ , where  $\tau$  is the flip, is again a comultiplication such that  $(A, \tau\Delta)$  is also a multiplier Hopf algebra. If  $A$  is a  $*$ -algebra, we call  $\Delta$  a comultiplication if it is also a  $*$ -homomorphism. A multiplier Hopf  $*$ -algebra is a  $*$ -algebra with a comultiplication, making it into a multiplier Hopf algebra.

An algebraic quantum group is a regular multiplier Hopf algebra with non-trivial invariant functionals (i.e., integrals).

Notice that we can use the Sweedler's notation for  $\Delta(x)$  when  $x \in A$ . The problem is that  $\Delta(x)$  is not in  $A \otimes A$  in general. By covering theory, we know however that  $\Delta(x)(1 \otimes y) \in A \otimes A$  for all  $x, y \in A$ . We can write (cf. [Dr-VD-Z, VD3])  $\Delta(x) = \sum x_1 \otimes x_2$  for this. Now, we know that there is an element  $e \in A$  such that  $y = ey$  and we can think of  $\Delta(x) = \sum x_1 \otimes x_2$  to stand for  $\Delta(x)(1 \otimes e)$ . Of course, this is still dependent on  $y$ . But we know that for several elements  $y$ , we can use the same  $e$ .

Let  $A$  be a regular multiplier Hopf algebra (i.e. with a bijective antipode). By a left  $A$ -module  $V$ , we always mean a unital module (sometimes, we also say that  $\triangleright$  is a left action of  $A$  on  $V$ ). This means that  $A \triangleright V = V$ . For all  $v \in V$ , we have an element  $e \in A$  such that  $e \triangleright v = v$ . Observe that for a unital algebra  $A$ , this condition is automatically satisfied. Similarly, a right  $A$ -module can be defined.

Let  $Q$  denote a regular multiplier Hopf algebra and  $A$  an associative algebra with or without identity. Assume that  $\triangleright$  is a left action of  $Q$  on  $A$  making  $A$  into a left  $Q$ -module algebra, i.e., we have  $x \triangleright (ab) = \sum (x_1 \triangleright a)(x_2 \triangleright b)$  for all  $x \in Q$  and  $a, b \in A$ . In this formula,  $x_1$  is covered by  $a$  (through the action) and  $x_2$  is covered by  $b$  (through the action). Similarly, assume that  $\triangleleft$  is a right action of  $Q$  on  $A$  making  $A$  into a right  $Q$ -module algebra, i.e., one has  $(ab) \triangleleft x = \sum (a \triangleleft x_1)(b \triangleleft x_2)$ , for all  $x \in Q$  and  $a, b \in A$ . Then  $A$  is called a  $Q$ -bimodule algebra, if it is a unital left  $Q$ -module and a unital right  $Q$ -module such that  $(x \triangleright a) \triangleleft y = x \triangleright (a \triangleleft y)$ , for all  $x, y \in Q$  and  $a \in A$ .

Suppose that  $\Gamma$  is a left coaction of  $Q$  on  $A$  making  $A$  a left  $Q$ -comodule algebra. More precisely,  $\Gamma : A \rightarrow M(Q \otimes A)$  is an injective homomorphism so that for all  $x \in Q$  and  $a \in A$ , we have  $\Gamma(a)(x \otimes 1)$  and  $(x \otimes 1)\Gamma(a)$  are in  $Q \otimes A$ . We use the Sweedler notation for these expressions, e.g.  $\Gamma(a)(x \otimes 1) = \sum a_{(-1)}x \otimes a_0$ . We say that the multiplier  $\Gamma(a)$  is covered by  $x \otimes 1$ . Now the further requirement makes sense:  $(\iota \otimes \Gamma)\Gamma(a) = (\Delta \otimes \iota)\Gamma(a)$  for all  $a \in A$ . In fact, the expression  $(\iota \otimes \Gamma)\Gamma(a) = \sum a_{(-2)} \otimes a_{(-1)} \otimes a_0$  makes sense when it is understood that  $\Delta(a_{(-1)})$  is replaced by  $a_{(-2)} \otimes a_{(-1)}$ . We have more or less the same rules for covering. We need to cover the factor  $a_{(-1)}$  and possibly also  $a_{(-2)}$  (and so on) by elements in  $A$ , left or right. In this case however, one cannot cover the factor  $a_0$ . Similarly, the right coaction  $\Upsilon$  of  $L$  on  $A$  can be defined. Then  $A$  is called a  $Q$ - $L$ -bicomodule algebra if

$$(1 \otimes 1 \otimes x)((\iota \otimes \Upsilon)(\Gamma(a)(y \otimes 1))) = ((\Gamma \otimes \iota)((1 \otimes x)\Upsilon(a)))(y \otimes 1)$$

for all  $x, y \in Q$  and  $a \in A$ .

Let  $Q, L$  be multiplier Hopf algebras. Denote the category of all left  $Q$ -module algebras by  ${}_Q\mathcal{MA}$ , denote by the category of all right  $L$ -module algebras  $\mathcal{MA}_L$ , and denote by  ${}_Q\mathcal{MA}_L$  the category of all  $Q$ - $L$ -bimodule algebras. Similarly, denote

the category of all left  $Q$ -comodule algebras by  ${}^Q\mathcal{MA}$ , denote by the category of all right  $L$ -module algebras  $\mathcal{MA}^L$ , and denote by  ${}^Q\mathcal{MA}^L$  the category of all  $Q$ - $L$ -bicomodule algebras.

Let  $Q$  and  $\widehat{Q}$  be a dual pair of algebraic quantum groups over  $\mathbb{C}$ . By Pontryagin duality, identifying  $\widehat{\widehat{Q}} = Q$ , we have a natural dual pairing  $Q \otimes \widehat{Q} \longrightarrow \mathbb{C}$  is written as

$$\langle x, \widehat{x} \rangle \in \mathbb{C}, \quad x \in Q, \text{ and } \widehat{x} \in \widehat{Q}.$$

Note that the antipode of  $Q^{op}$  and  $Q^{cop}$  is given by  $S^{-1}$  and the antipode of  $Q^{op,cop}$  by  $S$ . Also,  $\widehat{Q^{op}} = \widehat{Q}^{cop}$ ,  $\widehat{Q^{cop}} = \widehat{Q}^{op}$  and  $\widehat{Q^{op,cop}} = \widehat{Q}^{cop,op}$ .

Obviously, after a permutation of tensor factors  $A \otimes Q \longleftrightarrow Q \otimes A$  a left coaction of  $Q$  may always be viewed as a right coaction by  $Q^{cop}$  and vice versa.

Next, we recall that there is a one-to-one correspondence between right (left) coactions of  $Q$  on  $V$  and left (right) actions, respectively, of  $\widehat{Q}$  on  $V$  given for  $\widehat{x} \in \widehat{Q}$  and  $v \in V$  by

$$\widehat{x} \triangleright v = (\iota \otimes \widehat{x})\Upsilon(v)$$

and

$$v \triangleleft \widehat{x} = (\widehat{x} \otimes \iota)\Gamma(v).$$

As a particular example we recall the case  $V = Q$  with  $\Gamma = \Upsilon = \Delta$ . In this case we denote the associated left and right actions of  $\widehat{x} \in \widehat{Q}$  on  $y \in Q$  by  $\widehat{x} \blacktriangleright y$  and  $y \blacktriangleleft \widehat{x}$ , respectively, see [De].

We refer to [VD1-VD3] for the theory of multiplier Hopf algebras. For the use of the Sweedler notation in this setting, we refer to [Dr-VD] and [Dr-VD-Z]. For pairings of multiplier Hopf algebras, the main reference is [Dr-VD]. An important reference for this paper is of course [P-O] where the theory of a more general version of the so-called L-R-smash product for (quasi-)Hopf algebras is developed. Finally, for some related constructions with multiplier Hopf algebras, see [De, De-VD-W, De-VD] and [VD-VK, W, W-L].

## 2. Twisted tensor products

In this section, we will study the notions of a generalized twisted tensor product, a generalized twisted smash product and a generalized L-R-smash product. We also study some isomorphisms between them.

### 2.1 Generalized twisted tensor products

Let  $A, B$  be algebras and suppose that there are two  $k$ -linear maps  $R : B \otimes A \longrightarrow A \otimes B$  and  $T : A \otimes B \longrightarrow A \otimes B$  such that for  $a, c \in A, b, d \in B$

$$(2.1) \quad a_R \otimes (bd)_R = a_{Rr} \otimes b_r d_R \Leftrightarrow R(m_B \otimes id_A) = (id_A \otimes m_B)R_{12}R_{23};$$

$$(2.2) \quad (ac)_R \otimes b_R = a_R c_r \otimes b_{Rr} \Leftrightarrow R(id_B \otimes m_A) = (m_A \otimes id_B)R_{23}R_{12};$$

$$(2.3) \quad a_T \otimes (bd)_T = a_{Tt} \otimes b_T d_t \Leftrightarrow T(id_A \otimes m_B) = (id_A \otimes m_B)T_{13}T_{12};$$

$$(2.4) \quad (ac)_T \otimes b_T = a_T c_t \otimes b_{tT} \Leftrightarrow T(m_A \otimes id_B) = (m_A \otimes id_B)T_{13}T_{23};$$

$$(2.5) \quad a_T \otimes c_R \otimes d_{TR} = a_T \otimes c_R \otimes d_{RT} \Leftrightarrow R_{23}T_{12} = T_{13}R_{23}.$$

Here we write  $R(b \otimes a) = a_R \otimes b_R = a_r \otimes b_r$  and  $T(a \otimes b) = a_T \otimes b_T = a_t \otimes b_t$  for all  $a \in A$  and  $b \in B$ , and  $R_{12} = (R \otimes \iota) \in Hom_k(B \otimes A \otimes X, A \otimes B \otimes X)$ ,  $R_{23} = (\iota \otimes R) \in Hom_k(X \otimes B \otimes A, X \otimes A \otimes B)$ ,  $T_{12} = (T \otimes \iota) \in Hom_k(A \otimes B \otimes X, A \otimes B \otimes X)$ ,  $T_{13} \in Hom_k(A \otimes X \otimes B, A \otimes X \otimes B)$  works on the first and the third componets as  $T$  and  $T_{23} = (\iota \otimes T) \in Hom_k(X \otimes A \otimes B, X \otimes A \otimes B)$  for any  $k$ -vector space  $X$ .

One can define a *generalized L-R-twisted tensor product*  $A \#_{RT} B = A \otimes B$  as spaces with a new multiplication given by the formula:

$$m_{A \#_{RT} B} = (m_A \otimes m_B) \circ T_{14} \circ R_{23} \quad \text{or} \quad (a \# b)(c \# d) = a_T c_R \# b_R d_T \quad (2.1.1)$$

for all  $a, c \in A$  and  $b, d \in B$ . Here,  $T_{14} \in End_k(A \otimes B \otimes A \otimes B)$  works on the first and the forth componets as  $T$ .

**Proposition 2.1.1.** The above multiplication on  $A \#_{RT} B = A \otimes B$  is associative.

**Proof.** For all  $a, c, x \in A$  and  $b, d, y \in B$ . We compute

$$\begin{aligned} [(a \# b)(c \# d)](x \# y) &= (a_T c_R)_t x_r \# (b_R d_T)_r y_t \\ &\stackrel{(2.4)}{=} a_{Tt} c_{Rt'} x_r \# (b_R d_T)_r y_{t't} \\ &\stackrel{(2.1)}{=} a_{Tt} c_{Rt'} x_{rr'} \# b_{Rr'} d_{Tr} y_{t't}, \end{aligned}$$

and

$$\begin{aligned} (a \# b)[(c \# d)(x \# y)] &= a_T (c_t x_r)_R \# b_R (d_r y_t)_T \\ &\stackrel{(2.3)}{=} a_{Tt'} (c_t x_r)_R \# b_R d_{rT} y_{tt'} \\ &\stackrel{(2.2)}{=} a_{Tt'} c_{tR} x_{rR'} \# b_{RR'} d_{rT} y_{tt'} \\ &\stackrel{(2.5)}{=} a_{Tt'} c_{Rt} x_{rR'} \# b_{RR'} d_{rT} y_{tt'}. \end{aligned}$$

This completes the proof. ■

**Remark 2.1.2.** When  $T$  is an identity map, then a generalized L-R-twisted tensor product  $A \#_{RT} B = A \otimes B$  becomes a twisted tensor product  $A \#_R B = A \otimes B$ . As a space,  $A \#_R B = A \otimes B$  with the following product defined by the formula:

$$m_{A \#_R B} = (m_A \otimes m_B) \circ R_{23} \quad \text{or} \quad (a \# b)(c \# d) = ac_R \otimes b_R d$$

for all  $a, c \in A$  and  $b, d \in B$ . Here,  $R_{23} = \iota_A \otimes R \otimes \iota_B : A \otimes B \otimes A \otimes B \longrightarrow A \otimes A \otimes B \otimes B$ .

If  $A$  and  $B$  have a unit 1, we suppose that  $R$  satisfies  $R(1 \otimes a) = a \otimes 1$  and  $R(b \otimes 1) = 1 \otimes b$  for all  $a \in A$  and  $b \in B$ . Then  $1 \otimes 1$  is the unit of  $A \#_R B$ . Furthermore, the maps  $A \longrightarrow A \#_R B : a \mapsto a \otimes 1$  and  $B \longrightarrow A \#_R B : b \mapsto 1 \otimes b$  are algebra embeddings. For more details on the above results, we refer to [VD-VK].

**Definition 2.1.3.** Let  $A$  and  $B$  be two algebras without unit, but with non-degenerate products. Let  $R : B \otimes A \longrightarrow A \otimes B$  and  $T : A \otimes B \longrightarrow A \otimes B$  be two  $k$ -linear maps. Then one can define the following two  $k$ -linear maps  $R * T$  and  $T * R$  from  $A \otimes B \otimes A \otimes B$  to  $A \otimes B$ :

$$\begin{aligned}(R * T)(a \otimes x \otimes b \otimes y) &= (ab_R)_T \otimes (xy_T)_R; \\ (T * R)(b \otimes y \otimes a \otimes x) &= (b_T a)_R \otimes (y_R x)_T\end{aligned}$$

for all  $a, b \in A$  and  $x, y \in B$ .

**Proposition 2.1.4.** Let  $A$  and  $B$  be two algebras without unit, but with non-degenerate products. If  $R$  and  $T$  are bijective such that (1):  $(T * R)(x \otimes y \otimes a \otimes b) = 0$  implies that  $x \otimes y = 0$  and (2):  $(R * T)(a \otimes b \otimes x \otimes y) = 0$  implies that  $x \otimes y = 0$  for all  $a, x \in A$  and  $b, y \in B$ , then the product in  $A \#_{RT} B$  is non-degenerate.

**Proof.** Suppose that there is an element  $\sum a_i \# b_i \in A \#_{RT} B$  such that  $(\sum a_i \# b_i)(a \# b) = 0$  for all  $a \in A, b \in B$ . Then we have that

$$\sum a_{iT} a_r \# b_{ir} b_T = 0,$$

this implies (by Eq.(2.2)) that

$$\sum (a_{iTR^{-1}} a)_R \# b_{iR^{-1}} b_T = 0,$$

and by Eq.(2.3) and Eq.(2.5), one has that

$$\sum (a_{it^{-1}R^{-1}T} a)_R \# (b_{it^{-1}R^{-1}} b)_T = 0.$$

By the assumption (1), we have

$$\sum a_{it^{-1}R^{-1}} \# b_{it^{-1}R^{-1}} = 0.$$

Thus, we obtain that  $\sum a_i \# b_i = 0$ .

Similarly, by Eq.(2.1), (2.4), (2.5) and the assumption (2), one can show that  $(a \# b)(\sum a_i \# b_i) = 0$  for all  $a \in A, b \in B$  implies that  $\sum a_i \# b_i = 0$ . ■

**Corollary 2.1.5.** Let  $A$  and  $B$  be two algebras without unit, but with non-degenerate products.

(1) ([De, Proposition 1.1]) If  $R$  is bijective, then the product in  $A\#_R B$  is non-degenerate.

(2) If  $T$  is bijective, then the product in  $A\#_T B$  is non-degenerate.

**Proof.** (1) In this case,  $T = id_{A\otimes B}$ . Thus the conditions (1):  $(x_T a)_{R\otimes}(y_R b)_T = 0$  implies that  $x \otimes y = 0$  and (2):  $(ax_R)_T \otimes (by_T)_R = 0$  implies that  $x \otimes y = 0$  for all  $a, x \in A$  and  $b, y \in B$ , become that (1):  $(xa)_R \otimes y_R b = 0$  implies that  $x \otimes y = 0$  and (2):  $ax_R \otimes (by)_R = 0$  implies that  $x \otimes y = 0$  for all  $a, x \in A$  and  $b, y \in B$ . We now explain how these two conditions hold. In fact, for the condition (1), since the product on  $B$  is non-degenerate,  $(xa)_R \otimes y_R b = 0$  implies that  $(xa)_R \otimes y_R = 0$  and thus  $x \otimes y = 0$  since the product on  $B$  is non-degenerate. Similarly, the condition (2) holds too.

(2) In this case,  $R = id_{A\otimes B}$ . The check is similar to the one in the part (1). ■

**Theorem 2.1.6.** Let  $A$  and  $B$  be Hopf algebras with two  $k$ -linear maps  $R : B \otimes A \longrightarrow A \otimes B$  and  $T : A \otimes B \longrightarrow A \otimes B$  such that Eq.(2.1)-(2.5) hold and

$$(2.6) \quad a_R \otimes 1_R = a \otimes 1, \quad 1_R \otimes b_R = 1 \otimes b;$$

$$(2.7) \quad a_T \otimes 1_T = a \otimes 1, \quad 1_T \otimes b_T = 1 \otimes b;$$

$$(2.8) \quad (\varepsilon_A \otimes \varepsilon_B)R = (\varepsilon_B \otimes \varepsilon_A), \quad (\varepsilon_A \otimes \varepsilon_B)T = (\varepsilon_A \otimes \varepsilon_B);$$

$$(2.9) \quad (id_A \otimes \tau \otimes id_B)(\Delta_A \otimes \Delta_B)R = (R \otimes R)(id_B \otimes \tau \otimes id_A)(\Delta_B \otimes \Delta_A);$$

$$(2.10) \quad (id_A \otimes \tau \otimes id_B)(\Delta_A \otimes \Delta_B)T = (T \otimes T)(id_A \otimes \tau \otimes id_B)(\Delta_A \otimes \Delta_B).$$

Here we write  $R(b \otimes a) = a_R \otimes b_R = a_r \otimes b_r$  and  $T(a \otimes b) = a_T \otimes b_T = a_t \otimes b_t$  for all  $a \in A$  and  $b \in B$ .

Then  $(A\#_{RT} B, \Delta_{RT}, \varepsilon_{RT}, S_{RT})$  is a bialgebra, where the comultiplication,  $\Delta_{RT}$  and  $\varepsilon_{RT}$  are given as

$$\begin{aligned} (a\#b)(c\#d) &= a_T c_R \# b_R d_T, \\ \Delta_{RT} &= (id_A \otimes \tau \otimes id_B)(\Delta_A \otimes \Delta_B), \\ \varepsilon_{RT} &= \varepsilon_A \otimes \varepsilon_B, \end{aligned}$$

for all  $a, c \in A$  and  $b, d \in B$ .

Furthermore, if

$$(2.11) \quad \sum (S(a_1)_T a_2)_R \# (S(b_1)_R b_2)_T = \varepsilon_{RT}(a\#b)(1_A \# 1_B),$$

$$(2.12) \quad \sum (a_1 S(a_2)_R)_T \# (b_1 S(b_2)_T)_R = \varepsilon_{RT}(a\#b)(1_A \# 1_B),$$

then  $(A\#_{RT} B, \Delta_{RT}, \varepsilon_{RT}, S_{RT})$  is a Hopf algebra with  $S_{RT}$  given by

$$S_{RT} = TR(S_B \otimes S_A)\tau.$$

**Proof.** From Proposition 2.1.1, we know that  $A\#_{RT}B$  is an algebra with unity  $1_A\#1_B$ . Clearly  $\Delta_{RT}$  is coassociative and  $\varepsilon_{RT}$  is counitary.

Take  $a, c \in A$  and  $b, d \in B$ . We now prove that  $\Delta_{RT}$  is a homomorphism.

$$\begin{aligned}
& \Delta_{RT}[(a\#b)(c\#d)] \\
&= \Delta_{RT}(a_T c_R \# b_R d_T) \\
&= (id_A \otimes \tau \otimes id_B)(\Delta_A \otimes \Delta_B)(a_T c_R \# b_R d_T) \\
&= \sum (a_T c_R)_1 \# (b_R d_T)_1 \otimes (a_T c_R)_2 \# (b_R d_T)_2 \\
&= \sum a_{T1} c_{R1} \# b_{R1} d_{T1} \otimes a_{T2} c_{R2} \# b_{R2} d_{T2} \\
&\stackrel{(2.9)(2.10)}{=} \sum a_{1T} c_{1R} \# b_{1R} d_{1T} \otimes a_{2T} c_{2R} \# b_{2R} d_{2T} \\
&= \Delta_{RT}(a_T c_R \# b_R d_T) \Delta_{RT}(a_T c_R \# b_R d_T).
\end{aligned}$$

From the counitary property (2.8) we obtain that  $\varepsilon_{RT}$  is a homomorphism on  $A\#_{RT}B$ . Therefore,  $(A\#_{RT}B, \Delta_{RT}, \varepsilon_{RT}, S_{RT})$  is a bialgebra.

We show that  $m_{A\#_{RT}B}(S_{RT} \otimes id_{A\otimes B})(\Delta_{RT}(a\#b)) = \varepsilon_{RT}(a\#b)(1_A\#1_B)$ .

$$\begin{aligned}
& m_{A\#_{RT}B}(S_{RT} \otimes id_{A\otimes B})(\Delta_{RT}(a\#b)) \\
&= \sum S(a_1)_{RTt} a_{2r} \# S(b_1)_{RTt} b_{2t} \\
&\stackrel{(2.5)}{=} \sum S(a_1)_{RTt} a_{2r} \# S(b_1)_{RrT} b_{2t} \\
&\stackrel{(2.3)}{=} \sum S(a_1)_{RT} a_{2r} \# (S(b_1)_{Rr} b_2)_T \\
&\stackrel{(2.5)}{=} \sum S(a_1)_{TR} a_{2r} \# (S(b_1)_{Rr} b_2)_T \\
&\stackrel{(2.2)}{=} \sum (S(a_1)_T a_2)_R \# (S(b_1)_R b_2)_T \\
&\stackrel{(2.11)}{=} \varepsilon_{RT}(a\#b)(1_A\#1_B).
\end{aligned}$$

Similarly, by Eq.(2.1), (2.4), (2.5) and Eq.(2.12), we can get

$$\begin{aligned}
& m_{A\#_{RT}B}(id_{A\otimes B} \otimes S_{RT})(\Delta_{RT}(a\#b)) \\
&= \sum (a_1 S(a_2)_R)_T \# (b_1 S(b_2)_T)_R \\
&= \varepsilon_{RT}(a\#b)(1_A\#1_B).
\end{aligned}$$

Thus,  $(A\#_{RT}B, \Delta_{RT}, \varepsilon_{RT}, S_{RT})$  is a Hopf algebra. ■

**Remark 2.1.7.** In the  $(A\#_{RT}B, \Delta_{RT}, \varepsilon_{RT}, S_{RT})$ . We have

(1) when  $T$  is the identity map, i.e,  $T = id_{A\otimes B}$ , then the conditions (2.11) and (2.12) hold. In fact, we have

$$\sum (S(a_1)_T a_2)_R \# (S(b_1)_R b_2)_T$$



$$\begin{aligned}
&= \sum (S(a_1)a_2)_R \# S(b_1)_R b_2 \\
&= \sum \varepsilon(a) 1_R \# S(b_1)_R b_2 \\
&\stackrel{(2.6)}{=} \sum \varepsilon(a) 1 \# S(b_1) b_2 \\
&= \varepsilon_{RT}(a \# b)(1_A \# 1_B),
\end{aligned}$$

and so Eq.(2.11) is obtained. Similarly,

$$\sum (a_1 S(a_2)_R)_T \# (b_1 S(b_2)_T)_R = \varepsilon_{RT}(a \# b)(1_A \# 1_B).$$

In this case, we have already obtained the result of [De, Theorem 2.1].

(2) when  $R$  is the flip map, i.e.,  $\tau$ , then the conditions (2.11) and (2.12) hold too.

(3) When  $R$  and  $T$  are not trivial, we will give the example (see Example 2.3.1 (3)).

**Proposition 2.1.8.** Let  $A$  and  $B$  be Hopf  $*$ -algebras with two  $k$ -linear maps  $R : B \otimes A \longrightarrow A \otimes B$  and  $T : A \otimes B \longrightarrow A \otimes B$  such that  $R$  and  $T$  satisfy all the conditions of Theorem 2.0.6. If furthermore

$$(2.13) \quad (R(*_B \otimes *_A)\tau)^2 = id_A \otimes id_B,$$

$$(2.14) \quad (T(*_A \otimes *_B))^2 = id_A \otimes id_B,$$

then  $(\Delta_{RT}, \varepsilon_{RT}, S_{RT})$  is a Hopf  $*$ -algebra

**Proof.** By Theorem 2.1.6, one has that  $(A \#_{RT} B, \Delta_{RT}, \varepsilon_{RT}, S_{RT})$  is a Hopf algebra. By the conditions that  $R(*_B \otimes *_A)\tau$  and  $T(*_A \otimes *_B)$  are involutions on  $A \otimes B$ , we have that  $A \#_{RT} B$  is a  $*$ -algebra when  $(a \# b)^* = TR(b^* \otimes a^*)$ . In what follows, we check that  $\Delta_{RT}$  is a  $*$ -homomorphism on  $A \# B$ .

$$\begin{aligned}
\Delta_{RT}((a \# b)^*) &= \Delta_{RT}(TR(b^* \otimes a^*)) \\
&= (id_A \otimes \tau \otimes id_B)(\Delta_A \otimes \Delta_B)TR(b^* \otimes a^*) \\
&\stackrel{(2.10)(2.9)}{=} (TR \otimes TR)(id_B \otimes \tau \otimes id_A)(\Delta_B(b^*) \otimes \Delta_B(a^*)) \\
&= TR(b^*_1 \# a^*_1) \otimes TR(b^*_2 \# a^*_2) \\
&= (a_1 \# b_1)^* \otimes (a_2 \# b_2)^* \\
&= (\Delta_{RT}(a \# b))^*.
\end{aligned}$$

This finishes the proof. ■

**Proposition 2.1.9.** Let  $m \in M(A), n \in M(B), M \in M(A \otimes A)$  and  $N \in M(B \otimes B)$  be multipliers. Then

(1) we have that  $m \# n \in M(A \#_{RT} B)$  where  $m \# n$  is defined by

$$\begin{aligned}
(m \# n)_1(a \# x) &= m_1(a_{T^{-1}R^{-1}R})_T \# n_1(x_{T^{-1}R^{-1}T})_R; \\
(m \# n)_2(a \# x) &= m_2(a_{T^{-1}R^{-1}T})_R \# n_2(x_{T^{-1}R^{-1}R})_T
\end{aligned}$$

for all  $a \in A$  and  $x \in B$ .

(2) One has that  $M \# N \in M((A \#_{RT} B) \otimes (A \#_{RT} B))$  where  $M \# N$  is defined as follows:

$$\begin{aligned} (M \# N)_1((a \# x) \otimes (b \# y)) &= M_1^1(a_{T^{-1}R^{-1}R})_T \# N_1^1(x_{T^{-1}R^{-1}T})_R \\ &\quad \otimes M_1^2(b_{t^{-1}r^{-1}r})_t \# N_1^2(y_{t^{-1}r^{-1}t})_r; \\ (M \# N)_2((a \# x) \otimes (b \# y)) &= M_2^1(a_{T^{-1}R^{-1}T})_R \# N_2^1(x_{T^{-1}R^{-1}R})_T \\ &\quad \otimes M_2^2(b_{t^{-1}r^{-1}t})_r \# N_2^2(y_{t^{-1}r^{-1}r})_t \end{aligned}$$

for all  $a, b \in A$  and  $x, y \in B$ , where we write  $M_1 = M_1^1 \otimes M_1^2$  and  $N_1 = N_1^1 \otimes N_1^2$ .

**Proof.** (1) For all  $a, b \in A$  and  $x, y \in B$ .

$$\begin{aligned} &(a \# x)[(m \# n)_1(b \# y)] \\ &= a_t m_1(b_{T^{-1}R^{-1}R})_{TR'} \# x_{R'} n_1(y_{T^{-1}R^{-1}T})_{Rt} \\ &\stackrel{(2.2)}{=} (a_{tr^{-1}} m_1(b_{T^{-1}R^{-1}R})_T)_r \# x_{r^{-1}r} n_1(y_{T^{-1}R^{-1}T})_{Rt} \\ &\stackrel{(2.4)}{=} (a_{tr^{-1}T'^{-1}} m_1(b_{T^{-1}R^{-1}R}))_{Tr} \# x_{r^{-1}r} n_1(y_{T^{-1}R^{-1}TT'^{-1}})_{Rt} \\ &\stackrel{(2.5)}{=} (a_{tr^{-1}T'^{-1}} m_1(b_{T^{-1}R^{-1}R}))_{rT} \# x_{r^{-1}r} n_1(y_{T^{-1}R^{-1}TT'^{-1}})_{tR} \\ &= (m_2(a_{tr^{-1}T'^{-1}}) b_{T^{-1}R^{-1}R})_{rT} \# x_{r^{-1}r} n_1(y_{T^{-1}R^{-1}TT'^{-1}})_{tR} \\ &\stackrel{(2.2)}{=} (m_2(a_{tr^{-1}T'^{-1}})_r b_{T^{-1}R^{-1}Rr'})_T \# x_{r^{-1}rr'} n_1(y_{T^{-1}R^{-1}TT'^{-1}})_{tR} \\ &\stackrel{(2.1)}{=} (m_2(a_{tr^{-1}T'^{-1}})_r b_{T^{-1}R^{-1}R})_T \# (x_{r^{-1}r} n_1(y_{T^{-1}R^{-1}TT'^{-1}})_t)_{Rt} \\ &\stackrel{(2.3)}{=} (m_2(a_{t'^{-1}tr^{-1}T'^{-1}})_r b_{T^{-1}R^{-1}R})_T \# (x_{r^{-1}rt'^{-1}} n_1(y_{T^{-1}R^{-1}TT'^{-1}}))_{tR} \\ &= (m_2(a_{t'^{-1}tr^{-1}T'^{-1}})_r b_{T^{-1}R^{-1}R})_T \# (n_2(x_{r^{-1}rt'^{-1}}) y_{T^{-1}R^{-1}TT'^{-1}})_{tR} \\ &\stackrel{(2.5)(2.4)}{=} m_2(a_{t'^{-1}tr^{-1}T'^{-1}})_{rT} b_{T^{-1}R^{-1}RT''} \# (n_2(x_{r^{-1}rt'^{-1}}) y_{T^{-1}R^{-1}T''TT'^{-1}})_{Rt} \\ &\stackrel{(2.1)}{=} m_2(a_{t'^{-1}tr^{-1}T'^{-1}})_{rT} b_{T^{-1}R^{-1}Rr'T''} \# (n_2(x_{r^{-1}rt'^{-1}})_{r'} y_{T^{-1}R^{-1}T''TT'^{-1}R})_t \\ &\stackrel{(2.3)}{=} m_2(a_{t'^{-1}tt''r^{-1}T'^{-1}})_{rT} b_{T^{-1}R^{-1}Rr'T''} \# n_2(x_{r^{-1}rt'^{-1}})_{r't} y_{T^{-1}R^{-1}T''TT'^{-1}Rt''} \\ &\stackrel{(2.5)}{=} m_2(a_{T^{-1}R^{-1}T})_{Rt} b_r \# n_2(x_{T^{-1}R^{-1}R})_{Tr} y_t \\ &= (m_2(a_{T^{-1}R^{-1}T})_R \# n_2(x_{T^{-1}R^{-1}R})_T)(b \# y) \\ &= [(m \# n)_2(a \# x)](b \# y). \end{aligned}$$

(2) Similarly.

This finishes the proof. ■

**Definition 2.1.10.** Let  $A$  and  $B$  be multiplier Hopf algebras with two  $k$ -linear maps  $R : B \otimes A \longrightarrow A \otimes B$  and  $T : A \otimes B \longrightarrow A \otimes B$  as described above. One defines:

$$T_1^{A \#_{RT} B} = (id_{A \otimes B} \otimes R * T) \circ (\Delta_{A \otimes B} \otimes id_{A \otimes B}) \circ (R^{-1})_{34} \circ (T^{-1})_{34}$$

and

$$T_2^{A\#_{RT}B} = (T * R \otimes id_{A \otimes B}) \circ (id_{A \otimes B} \otimes \Delta_{A \otimes B}) \circ (T^{-1})_{12} \circ (R^{-1})_{12}.$$

We now prove that these candidates  $T_1^{A\#_{RT}B}$  and  $T_2^{A\#_{RT}B}$  defined by  $R * T$  and  $T * R$ , respectively can be used to define a good comultiplication.

**Proposition 2.1.11.** Take the notations as above. For all  $a, b, c \in A$  and  $x, y, z \in B$ , define

$$\Delta_{RT}(a\#x)((b\#y) \otimes (c\#z)) = T_1^{A\#_{RT}B}((a\#x) \otimes (b\#y))((c\#z) \otimes (1\#1))$$

and

$$((b\#y) \otimes (c\#z))\Delta_{RT}(a\#x) = ((1\#1) \otimes (c\#z))T_2^{A\#_{RT}B}((a\#x) \otimes (b\#y)).$$

Then  $\Delta_{RT}(a\#x)$  is a two-sided multiplier of  $(A\#_{RT}B) \otimes (A\#_{RT}B)$ , where  $\Delta_{RT}(a\#x) = \Delta(a)\#\Delta(x)$ . Furthermore,  $\Delta_{RT}$  is coassociative on  $A\#_{RT}B$ .

**Proof.** For all  $a, b \in A$  and  $x, y \in B$ , we compute.

$$\begin{aligned} & T_1^{A\#_{RT}B}((a\#x) \otimes (b\#y)) \\ &= (id_{A \otimes B} \otimes R * T) \circ (\Delta_{A \otimes B} \otimes id_{A \otimes B}) \circ (R^{-1})_{34} \circ (T^{-1})_{34}((a\#x) \otimes (b\#y)) \\ &= (id_{A \otimes B} \otimes R * T) \circ (\Delta_{A \otimes B} \otimes id_{A \otimes B})((a\#x) \otimes (b_{t^{-1}R^{-1}}\#y_{t^{-1}R^{-1}})) \\ &= \sum (a_1\#x_1) \otimes (R * T)((a_2\#x_2) \otimes (b_{t^{-1}R^{-1}}\#y_{t^{-1}R^{-1}})) \\ &= \sum (a_1\#x_1) \otimes (a_2b_{t^{-1}R^{-1}R})_T\#(x_2y_{t^{-1}R^{-1}T})_R \\ &\stackrel{(2.5)}{=} \sum (a_1\#x_1) \otimes (a_2b_{R^{-1}Rt^{-1}})_T\#(x_2y_{t^{-1}TR^{-1}})_R \\ &\stackrel{(2.1)}{=} \sum (a_1\#x_1) \otimes (a_2b_{R^{-1}Rrt^{-1}})_T\#x_{2r}y_{t^{-1}TR^{-1}R} \\ &= \sum (a_1\#x_1) \otimes (a_2b_{rt^{-1}})_T\#x_{2r}y_{t^{-1}T} \\ &\stackrel{(2.4)}{=} \sum (a_1\#x_1) \otimes a_{2T}b_{rt^{-1}t}\#x_{2r}y_{t^{-1}tT} \\ &= \sum (a_1\#x_1) \otimes a_{2T}b_r\#x_{2r}y_T \\ &= \Delta_{RT}(a\#x)((1\#1) \otimes (b\#y)). \end{aligned}$$

Similarly, by Eq.(2.2), (2.3) and (2.5) we have

$$T_2^{A\#_{RT}B}((a\#x) \otimes (b\#y)) = ((a\#x) \otimes (1\#1))\Delta_{RT}(b\#y).$$

Firstly, for  $a \in A, x \in B$ , we prove that  $\Delta_{RT}(a\#x)$  is a two-sided multiplier of  $(A\#_{RT}B) \otimes (A\#_{RT}B)$ . In fact, notice that  $\Delta_{RT}(a\#x) = \Delta_A(a)\#\Delta_B(x)$  and recall from Proposition 2.1.8 that for  $\Delta_A(a) \in M(A \otimes A)$  and  $\Delta_B(x) \in M(B \otimes B)$ , we

can form the multiplier  $\Delta_A(a)\#\Delta_B(x) \in M((A\#_{RT}B) \otimes (A\#_{RT}B))$ . We can easily check that  $\Delta_{RT}(a\#x) = \Delta_A(a)\#\Delta_B(x)$ . It is clear that in the case that  $A$  and  $B$  are Hopf algebras,  $\Delta_{RT}(a\#x) = (id_A \otimes \tau \otimes id_B)(\Delta_A(a) \otimes \Delta_B(x))$ .

Then in order to check that  $\Delta_{RT}$  is coassociative on  $A\#_{RT}B$ , we have to prove that

$$(T_2^{A\#_{RT}B} \otimes id)(id \otimes T_1^{A\#_{RT}B}) = (id \otimes T_1^{A\#_{RT}B})(T_2^{A\#_{RT}B} \otimes id).$$

This calculations are straightforward by making use of the expressions

$$T_1^{A\#_{RT}B} = (id_{A \otimes B} \otimes R * T) \circ (\Delta_{A \otimes B} \otimes id_{A \otimes B}) \circ (R^{-1})_{34} \circ (T^{-1})_{34}$$

and

$$T_2^{A\#_{RT}B} = (T * R \otimes id_{A \otimes B}) \circ (id_{A \otimes B} \otimes \Delta_{A \otimes B}) \circ (T^{-1})_{12} \circ (R^{-1})_{12}$$

and the coassociativity of  $\Delta_A$  and  $\Delta_B$ .

This completes the proof. ■

We now have the main result of this section as follows.

**Theorem 2.1.12.** Let  $A$  and  $B$  be multiplier Hopf algebras with two  $k$ -linear maps  $R : B \otimes A \longrightarrow A \otimes B$  and  $T : A \otimes B \longrightarrow A \otimes B$  such that Eq.(2.1)-(2.5) hold. If  $R$  and  $T$  are bijective such that (1):  $(T * R)(x \otimes y \otimes a \otimes b) = 0$  implies that  $x \otimes y = 0$  and (2):  $(R * T)(a \otimes b \otimes x \otimes y) = 0$  implies that  $x \otimes y = 0$  for all  $a, x \in A$  and  $b, y \in B$ , and

$$(2.15) \quad (\Delta_{RT} \circ R)(x \otimes a) = ((1 \otimes 1)\#\Delta_B(x))(\Delta_A(a)\#(1 \otimes 1)) \text{ for } a \in A, x \in B;$$

$$(2.16) \quad \Delta_{RT} \circ T = (T \otimes T) \circ \Delta_{RT};$$

$$(2.17) \quad (T * R) \circ (S_A \otimes S_B \otimes id_{A \otimes B}) \circ \Delta_{RT} = (1\#1)\varepsilon_{RT},$$

$$(2.18) \quad (R * T) \circ (id_{A \otimes B} \otimes S_A \otimes S_B) \circ \Delta_{RT} = (1\#1)\varepsilon_{RT},$$

then  $(A\#_{RT}B, \Delta_{RT}, \varepsilon_{RT}, S_{RT})$  is a multiplier Hopf algebra, where the multiplication,  $\Delta_{RT}$ ,  $\varepsilon_{RT}$  and  $S_{RT}$  are given as

$$\begin{aligned} (a\#b)(c\#d) &= a_T c_R \# b_R d_T, \\ \Delta_{RT} &= (id_A \otimes \tau \otimes id_B)(\Delta_A \otimes \Delta_B), \\ \varepsilon_{RT} &= \varepsilon_A \otimes \varepsilon_B, \\ S_{RT} &= T \circ R \circ (S_B \otimes S_A)\tau \end{aligned}$$

for all  $a, c \in A$  and  $b, d \in B$ .

**Proof.** We will finish the proof with the following steps:

(1) From the conditions Eq.(2.1)-(2.5) and Proposition 2.1.4, and the bijectivities of  $T$  and  $R$  we have that  $A\#_{RT}B$  is an associative algebra with non-degenerate product.

(2) From Proposition 2.1.10 we conclude that  $\Delta_{RT}$  is coassociative on  $A\#_{RT}B$ .

(3) We prove that  $\Delta_{RT} : (A\#_{RT}B) \longrightarrow M((A\#_{RT}B) \otimes (A\#_{RT}B))$  is a homomorphism. For all  $a, b \in A$  and  $x, y \in B$ , we have

$$\begin{aligned}
& \Delta_{RT}[(a\#x)(b\#y)] = \Delta_{RT}(a_T b_R \# x_R y_T) \\
&= \Delta_A(a_T b_R) \# \Delta_B(x_R y_T) \\
&= (\Delta_A(a_T) \# (1 \otimes 1)) (\Delta_A(b_R) \# \Delta_B(x_R)) ((1 \otimes 1) \# \Delta_B(y_T)) \\
&= (\Delta_A(a_T) \# (1 \otimes 1)) (\Delta_{RT} \circ R)(x \otimes b) ((1 \otimes 1) \# \Delta_B(y_T)) \\
&\stackrel{(2.15)}{=} (\Delta_A(a_T) \# (1 \otimes 1)) ((1 \otimes 1) \# \Delta_B(x)) (\Delta_A(b) \# (1 \otimes 1)) ((1 \otimes 1) \# \Delta_B(y_T)) \\
&\stackrel{(2.16)}{=} (\Delta_A(a) \# \Delta_B(x)) (\Delta_A(b) \# \Delta_B(y)) \\
&= \Delta_{RT}(a\#x) \Delta_{RT}(b\#y).
\end{aligned}$$

(4) Define the counit  $\varepsilon_{RT}$  on  $A\#_{RT}B$  by  $\varepsilon_{RT} = \varepsilon_A \otimes \varepsilon_B$ . We have to prove that for all  $a, b \in A$  and  $x, y \in B$

- (i)  $(\varepsilon_{RT} \otimes id_A \otimes id_B) \Delta_{RT}(a\#x) ((1\#1) \otimes (b\#y)) = (a\#x)(b\#y);$
- (ii)  $(id_A \otimes id_B \otimes \varepsilon_{RT}) ((a\#x) \otimes (1\#1)) \Delta_{RT}(b\#y) = (a\#x)(b\#y).$

We prove (i), the proof of (ii) is similar. From Proposition 2.1.10 we have that for  $a, b \in A$  and  $x, y \in B$

$$\begin{aligned}
& \Delta_{RT}(a\#x) ((1\#1) \otimes (b\#y)) \\
&= T_1^{A\#_{RT}B} ((a\#x) \otimes (b\#y)) \\
&= \sum (a_1 \otimes x_1 \otimes \sum_{ij} (a_2 b_i^j)_R \# (x_2 y_i^j)_R)
\end{aligned}$$

where  $T^{-1}(b \otimes y) = \sum_i y_i \otimes b_i$  and  $R^{-1}(\sum_i y_i \otimes b_i) = \sum_{ij} b_i^j \otimes y_i^j$ .  
So,

$$\begin{aligned}
& (\varepsilon_{RT} \otimes id_A \otimes id_B) \Delta_{RT}(a\#x) ((1\#1) \otimes (b\#y)) \\
&= (\varepsilon_{RT} \otimes id_A \otimes id_B) \sum (a_1 \otimes x_1 \otimes \sum_{ij} (a_2 b_i^j)_R \# (x_2 y_i^j)_R) \\
&\stackrel{(2.4)}{=} \sum_{ij} a_T b_i^j \# (x y_i^j)_R \\
&\stackrel{(2.1)}{=} \sum_{ij} a_T b_i^j \# x_r y_i^j \# x_r y_i^j \\
&\stackrel{(2.5)}{=} \sum_i a_T b_{ir} \# x_r y_{iT} \\
&\stackrel{(2.5)}{=} a_T b_r \# x_r y_T \\
&= (a\#x)(b\#y).
\end{aligned}$$

(5) Because  $T_1^{A\#_{RT}B}$  is surjective and  $\Delta_{RT}$  is a homomorphism,  $\varepsilon_{RT}$  is a homomorphism. This can be proved in a similar way as in [VD1, Lemma 3.5].

(6) On  $A \#_{RT} B$ , define the antipode  $S_{RT} = T \circ R \circ (S_B \otimes S_A)\tau$  which is an invertible map. We have to prove that for all  $a, b \in A$  and  $x, y \in B$

- (i)  $m_{A \#_{RT} B}(S_{RT} \otimes id_A \otimes id_B)\Delta_{RT}(a \# x)((1 \# 1) \otimes (b \# y)) = \varepsilon_{RT}(a \# x)(b \# y)$ ;
- (ii)  $m_{A \#_{RT} B}(id_A \otimes id_B \otimes S_{RT})((a \# x) \otimes (1 \# 1))\Delta_{RT}(b \# y) = (a \# x)\varepsilon_{RT}(b \# y)$ .

We prove (i), the proof of (ii) is similar. In fact, start again from the following equation:

$$\Delta_{RT}(a \# x)((1 \# 1) \otimes (b \# y)) = \sum (a_1 \otimes x_1 \otimes \sum_{ij} (a_2 b_i^j)_R)_T \# (x_2 y_i^j)_T)_R$$

where  $T^{-1}(b \otimes y) = \sum_i y_i \otimes b_i$  and  $R^{-1}(\sum_i y_i \otimes b_i) = \sum_{ij} b_i^j \otimes y_i^j$ .

Then we have

$$\begin{aligned} & m_{A \#_{RT} B}(S_{RT} \otimes id_A \otimes id_B)\Delta_{RT}(a \# x)((1 \# 1) \otimes (b \# y)) \\ = & m_{A \#_{RT} B}(S_{RT} \otimes id_A \otimes id_B) \sum (a_1 \otimes x_1 \otimes \sum_{ij} (a_2 b_i^j)_R)_T \# (x_2 y_i^j)_T)_R \\ = & \sum_{ij} \sum S(a_1)_{R'T't} (a_2 b_i^j)_R)_T \# S(b_1)_{R'T'r} (x_2 y_i^j)_T)_R \\ = & \sum ((S(a_1)_T a_2)_R \# (S(b_1)_R b_2)_T) (b \# y) \quad \text{by (2.1)-(2.5)} \\ \stackrel{(2.17)}{=} & \varepsilon_{RT}(a \# x)(b \# y). \end{aligned}$$

(7) Because  $T_1^{A \#_{RT} B}$  is surjective and  $\Delta_{RT}$  is a homomorphism,  $S_{RT}$  is a anti-homomorphism. The proof is similar to the proof of [VD1, Lemma 4.4].

It follows from [VD2, Proposition 2.9] that we now conclude that  $(A \#_{RT} B, \Delta_{RT}, \varepsilon_{RT}, S_{RT})$  is a (regular) multiplier Hopf algebra. ■

**Proposition 2.1.13.** Let  $A$  and  $B$  be multiplier Hopf  $*$ -algebras with two  $k$ -linear maps  $R : B \otimes A \rightarrow A \otimes B$  and  $T : A \otimes B \rightarrow A \otimes B$  such that  $R$  and  $T$  satisfy all the conditions of Theorem 2.1.12. If furthermore

$$(2.19) \quad (R(*_B \otimes *_A)\tau)^2 = id_A \otimes id_B,$$

$$(2.20) \quad (T(*_A \otimes *_B))^2 = id_A \otimes id_B,$$

then  $(A \#_{RT} B, \Delta_{RT}, \varepsilon_{RT}, S_{RT})$  is a multiplier Hopf  $*$ -algebra.

**Proof.** Straightforward. ■

**Proposition 2.1.14.** Let  $A$  and  $B$  be multiplier Hopf algebras as in Theorem 2.1.12. Let  $\psi_A$  (resp.  $\psi_B$ ) be a right integral on  $A$  (resp.  $B$ ). Then  $\psi_A \otimes \psi_B$  is a right integral on the multiplier Hopf algebra  $(A \#_{RT} B, \Delta_{RT}, \varepsilon_{RT}, S_{RT})$ .

**Proof.** For all  $a, b \in A$  and  $x, y \in B$ ,

$$\begin{aligned} & ((\psi_A \otimes \psi_B \otimes id_{A \otimes B})\Delta_{RT}(a \# b))(x \# y) \\ = & (\psi_A \otimes \psi_B \otimes id_{A \otimes B})(\Delta_{RT}(a \# b)((1 \# 1) \otimes (x \# y))) \end{aligned}$$

$$\begin{aligned}
&= (\psi_A \otimes \psi_B \otimes id_{A \otimes B}) T_1^{A \#_{RT} B} ((a \# x) \otimes (b \# y)) \\
&= \sum (\psi_A \otimes \psi_B \otimes id_{A \otimes B}) (a_1 \# x_1) \otimes (a_2 b_{R^{-1} R t^{-1}})_T \# (x_2 y_{t^{-1} T R^{-1}})_R \\
&= \psi_A(a) \psi_B(x) (b_{R^{-1} R t^{-1}})_T \# (y_{t^{-1} T R^{-1}})_R \\
&= \psi_A(a) \psi_B(x) (b \# y).
\end{aligned}$$

This finishes the proof. ■

**Proposition 2.1.15.** Let  $A$  and  $B$  be multiplier Hopf algebras as in Theorem 2.1.12. Let  $\varphi_A$  (resp.  $\varphi_B$ ) be a left integral on  $A$  (resp.  $B$ ) with associated modular element  $\delta_A$  (resp.  $\delta_B$ ). Then the multiplier  $\delta_A \# \delta_B$  is the modular element in  $M(A \#_{RT} B)$  associated to  $\varphi_A \otimes \varphi_B$ .

**Proof.** Recall from [VD2] that the modular element  $\delta_A$  in  $M(A)$  for a multiplier Hopf algebra  $A$  is given by  $(\varphi_A \otimes id) \Delta(a)$  when  $\varphi_A(a) = 1$ . Now, for the multiplier Hopf algebra  $(A \#_{RT} B, \Delta_{RT}, \varepsilon_{RT}, S_{RT})$ , we have that modular element  $\delta_{A \#_{RT} B}$  associated to  $\varphi_A \otimes \varphi_B$  is given as

$$\delta_{A \#_{RT} B} = (\varphi_A \otimes \varphi_B \otimes id_{A \otimes B}) \Delta_{RT}(a \# b).$$

We claim that  $\delta_{A \#_{RT} B} = \varphi_A \otimes \varphi_B$

For all  $a, b \in A$  and  $x, y \in B$ , we have

$$\begin{aligned}
\delta_{A \#_{RT} B}(b \# y) &= (\varphi_A \otimes \varphi_B \otimes id_{A \otimes B}) (\Delta_{RT}(a \# b) ((1 \# 1) \otimes (x \# y))) \\
&= (\varphi_A \otimes \varphi_B \otimes id_{A \otimes B}) T_1^{A \#_{RT} B} ((a \# x) \otimes (b \# y)) \\
&= \sum (\varphi_A \otimes id_{A \otimes B}) (a_1 \otimes (a_2 b_{R^{-1} R t^{-1}})_T \# (\delta_B y_{t^{-1} T R^{-1}})_R) \\
&= (\delta_A b_{R^{-1} R t^{-1}})_T \# (\delta_B y_{t^{-1} T R^{-1}})_R \\
&\stackrel{(2.1)}{=} (\delta_A b_{R^{-1} R r t^{-1}})_T \# \delta_{B r} y_{t^{-1} T R^{-1} R} \\
&= (\delta_A b_{r t^{-1}})_T \# \delta_{B r} y_{t^{-1} T} \\
&\stackrel{(2.4)}{=} \delta_{A T} b_{r t^{-1} t} \# \delta_{B r} y_{t^{-1} t T} \\
&= \delta_{A T} b_r \# \delta_{B r} y_T \\
&= (\delta_A \# \delta_B)(b \# y).
\end{aligned}$$

This completes the proof. ■

## 2.2 Generalized smash products

Let  $Q$  be a regular multiplier Hopf algebra and let  $A$  be a right  $Q$ -module algebra and  $B$  a right  $Q$ -comodule algebra. Then there are  $k$ -linear maps  $R : A \otimes B \rightarrow B \otimes A, a \otimes b \mapsto \sum b_0 \otimes (a \triangleleft b_{(1)})$  for  $a \in A$  and  $b \in B$ . By Remark 2.1.2, we may define a right smash product  $B \#_r^Q A$  as  $k$ -vector space with multiplication

$$(b' \# a')(b \# a) = \sum b' b_0 \# (a' \triangleleft b_{(1)}) a \quad (2.2.1)$$

for  $a, a' \in A, b, b' \in B$ , where we use the notation  $b\#a$  in place of  $b \otimes a$  to emphasize the new algebraic structure.

Let  $Q$  be a regular multiplier Hopf algebra and let  $A$  be a left  $Q$ -module algebra and  $B$  a left  $Q$ -comodule algebra. Then there are another  $k$ -linear maps  $R : B \otimes A \longrightarrow A \otimes B, b \otimes a \mapsto \sum b_{(-1)} \triangleright a \otimes b_0$  for  $a \in A$  and  $b \in B$ . Similarly, by Remark 2.1.2,  $A\#_l^Q B$  denotes the associative algebra structure on  $A \otimes B$  given by

$$(a\#b)(a'\#b') = \sum a(b_{(-1)} \triangleright a')\#b_0b' \quad (2.2.2)$$

for  $a, a' \in A, b, b' \in B$ , containing  $A$  and  $B$  as subalgebras.

We have obvious embeddings of  $A$  and  $B$  in the multiplier algebra of  $B\#_r^Q A$ . And we note that  $B\#_r^Q A = BA = AB$  with  $ab = \sum b_0(a \triangleleft b_{(1)})$  for  $a \in A, b \in B$ . Similar statements hold in  $A\#_l^Q B$ .

**Example 2.2.1** (1) Given a left coaction  $\Gamma : A \longrightarrow M(Q \otimes A)$  of  $Q$  on  $A$  with dual right  $\widehat{Q}$ -action  $\triangleleft$ . One has the right smash product  $\widehat{Q}\#_r^{\widehat{Q}} A$  to be the vector space  $\widehat{Q} \otimes A$  with associative algebra structure given for  $\widehat{x}, \widehat{y} \in \widehat{Q}$  and  $a, b \in A$  by

$$(\widehat{x}\#a)(\widehat{y}\#b) = \sum \widehat{x}\widehat{y}_1\#(a \triangleleft \widehat{y}_2)b$$

(2) Similarly if  $\Upsilon : A \longrightarrow M(A \otimes Q)$  is a right coaction of  $Q$  on  $A$  with dual left  $\widehat{Q}$ -action  $\triangleright$ , then the multiplication of the algebra  $A\#_l^{\widehat{Q}} \widehat{Q}$  is given by

$$(a\#\widehat{x})(b\#\widehat{y}) = \sum a(\widehat{x}_1 \triangleright b)\#\widehat{x}_2\widehat{y}$$

for  $\widehat{x}, \widehat{y} \in \widehat{Q}$  and  $a, b \in A$

We remark that any right smash product  $\widehat{Q}\#_r^{\widehat{Q}} A$  can be identified with an associated left smash product  $A\#_l^{\widehat{Q}^{cop}} \widehat{Q}^{cop}$ , where  $\Upsilon : A \longrightarrow M(A \otimes Q^{op})$  is the right coaction given by

$$\Upsilon = (\iota \otimes S^{-1}) \circ (\tau_{Q,A}) \circ \Gamma \quad (2.2.3)$$

with  $\tau_{Q,A} : Q \otimes A \longrightarrow A \otimes Q$  being the permutation of tensor factors. In fact we have

**Lemma 2.2.2.** Let  $\Gamma : A \longrightarrow M(Q \otimes A)$  and  $\Upsilon : A \longrightarrow M(A \otimes Q^{op})$  be a pair of left and right coactions, respectively, related by (2.2.3). Then we have

$$\widehat{Q}\#_r^{\widehat{Q}} A \cong A\#_l^{\widehat{Q}^{cop}} \widehat{Q}^{cop}.$$

**Proof.** Let  $\triangleright : \widehat{Q}^{cop} \otimes A \longrightarrow A$  and  $\triangleleft : A \otimes \widehat{Q} \longrightarrow A$  be the left and right actions dual to  $\Gamma$  and  $\Upsilon$ , respectively. The equation (2.2.3) is equivalent to  $\widehat{x}\triangleright a = a\triangleleft \widehat{S}^{-1}(\widehat{x})$  for all  $\widehat{x} \in \widehat{Q}$  and  $a \in A$ .



Define a linear map  $f : A \#_l^{\widehat{Q}^{cop}} \widehat{Q}^{cop} \longrightarrow \widehat{Q} \#_r^{\widehat{Q}} A$  as

$$f(a \# \widehat{x}) = \sum \widehat{x}_1 \# (a \triangleleft \widehat{x}_2).$$

It is easy to check that its inverse is given by

$$f^{-1}(\widehat{x} \# a) = \sum (\widehat{x}_1 \triangleright a) \# \widehat{x}_2$$

for  $\widehat{x} \in \widehat{Q}$  and  $a \in A$ .

The remaining thing is straightforward. ■

Let  $Q, L$  be Hopf algebras and let  $A$  be a right  $Q$ -module algebra and  $B$  a right  $L$ -comodule algebra. Let  $f : L \longrightarrow Q$  be a bialgebra map. Then there are a  $k$ -linear maps  $R : A \otimes B \longrightarrow B \otimes A, a \otimes b \mapsto \sum b_0 \# (a \triangleleft f(b_{(1)}))$  for  $a \in A$  and  $b \in B$ . By Remark 2.1.2, we can define a right smash product  $B \#_r^{(Q,L,f)} A$  as  $k$ -vector space with multiplication

$$(b' \# a')(b \# a) = \sum b' b_0 \# (a' \triangleleft f(b_{(1)})) a \quad (2.2.4)$$

for  $a, a' \in A, b, b' \in B$ , where we use the notation  $b \# a$  in place of  $b \otimes a$  to emphasize the new algebraic structure.

Then the proof of the following proposition is straightforward.

**Proposition 2.2.3.** With the above notation.  $B \#_r^{(Q,L,f)} A$  is an associative algebra.

Similarly, let  $A$  be a left  $Q$ -module algebra and  $B$  a left  $L$ -comodule algebra. Let  $f : L \longrightarrow Q$  be a bialgebra map. Then there are a  $k$ -linear maps  $R : B \otimes A \longrightarrow A \otimes B, b \otimes a \mapsto \sum (f(b_{(-1)}) \triangleright a) \# b_0$  for  $a \in A$  and  $b \in B$ . By Remark 2.1.2,  $A \#_l^{(Q,L,f)} B$  denotes the associative algebra structure on  $A \otimes B$  given by

$$(a \# b)(a' \# b') = \sum a(f(b_{(-1)}) \triangleright a') \# b_0 b' \quad (2.2.5)$$

for  $a, a' \in A, b, b' \in B$ , containing  $A$  and  $B$  as subalgebras.

### 2.3 Generalized twisted smash products

Let  $Q$  be a regular multiplier Hopf algebra. Let  $A$  be a  $Q$ -bimodule algebra, i.e.,  $A \in {}_Q \mathcal{MA}_Q$  and let  $B$  be a  $Q$ -bicomodule algebra, i.e.,  $B \in {}^Q \mathcal{MA}^Q$ . In this section, we consider an algebra  $P$  that is a generalized smash product of  $A$  and  $B$ . The construction has probably been studied in [W] for Hopf algebras but not yet for multiplier Hopf algebras. However, the results and the arguments are very

similar to the theory of smash products as developed in [Dr-VD-Z]. Therefore, in the following proposition, we do not give all the details. We concentrate on the correct statements and briefly indicate how things are proven.

Define  $B \star_r^Q A = B \otimes A$  as  $k$ -vector space with multiplication

$$(b' \star a')(b \star a) = \sum b' b_0 \star (S^{-1}(b_{(-1)}) \triangleright a' \triangleleft b_{(1)}) a \quad (2.3.1)$$

for  $a, a' \in A, b, b' \in B$ .

Similarly  $A \star_l^Q B$  denotes the associative algebra structure on  $A \otimes B$  given by

$$(a \star b)(a' \star b') = \sum a(b_{(-1)} \triangleright a' \triangleleft S^{-1}(b_{(1)})) \star b_0 b' \quad (2.3.2)$$

for  $a, a' \in A, b, b' \in B$ .

**Proposition 2.3.1**  $B \star_r^Q A$  and  $A \star_l^Q B$  as above are associative algebras.

**Proof.** For  $B \star_r^Q A$ , the twist map is given by the formula:  $R : A \otimes B \rightarrow B \otimes A, a \otimes b = \sum b_0 \otimes (S^{-1}(b_{(-1)}) \triangleright a \triangleleft b_{(1)})$  for  $a \in A$  and  $b \in B$ . By Remark 2.1.2,  $B \star_r^Q A$  is an associative algebra.

Similarly for  $A \star_l^Q B$  and the proof is finished. ■

The algebra  $B \star_r^Q A$  (resp.  $A \star_l^Q B$ ) is called a generalized right (resp. left) smash product, which is denoted by  $P$  (resp.  $U$ ). Just as in the case of smash products, we have obvious embeddings of  $A$  and  $B$  in the multiplier algebra of  $P$  (resp.  $U$ ) and if we identify these two algebras with their images in  $M(P)$  (resp.  $M(U)$ ), we see that  $P$  (resp.  $U$ ) is the linear span of elements  $ab$  (resp.  $ba$ ) with  $a \in A$  and  $b \in B$  and that we have the commutation rules

- i)  $A$  and  $B$  commute,
- ii)  $ab = \sum b_0 (S^{-1}(b_{(-1)}) \triangleright a \triangleleft b_{(1)})$   
(resp.  $ba = \sum (b_{(-1)} \triangleright a \triangleleft S^{-1}(b_{(1)})) b_0$ ), for  $a \in A, b \in B$ .

Therefore we can view  $P$  (resp.  $U$ ) as the algebra generated by  $A$  and  $B$  subject to these commutation rules.

**Example 2.3.2.** (1) This construction reduces to well-know constructions in the following three special situations. If the multiplier Hopf algebra  $Q$  is trivial, then we obtain for  $P$  simply the tensor product algebra  $A \otimes B$ . If the left action and coaction of  $Q$  on  $A$  and  $B$  are trivial, respectively, we obtain the right smash product  $B \#_r^Q A$ .

(2) Let  $Q$  denote an algebraic quantum group. Then  $\widehat{Q}$  is the  $Q$ -bimodule algebra. Let  $B$  be a  $Q$ -bicomodule algebra. Then we have two generalized twisted smash products  $B \star_r^Q \widehat{Q}$  and  $\widehat{Q} \star_l^Q B$ .

(3) Let  $Q$  denote a regular multiplier Hopf algebra with antipode  $S^2 = \iota$  and  $A$  a  $Q$ -bimodule algebra. Then the generalized twisted smash product  $A \star_l^Q Q$  is

isomorphic to an ordinary smash product  $A \# Q$ , where the left  $Q$ -action on  $Q$  is now given by  $x \rightarrow a = \sum x_1 \triangleright a S(x_2)$ , for all  $a \in A$  and  $x \in Q$ . Recall that in the original paper [Dr-V-Z], one developed the theory for left actions.

#### 2.4 Generalized L-R-smash products

The L-R-smash product was introduced and studied in a series of papers [P-O], with motivation and examples coming from the theory of deformation quantization. However, we will study the case slightly different from [P-O].

Let  $Q$  be a multiplier Hopf algebra,  $A$  a  $Q$ -bimodule algebra, i.e.,  $A \in {}_Q\mathcal{MA}_Q$  and  $B$  a  $Q$ -bicomodule algebra, i.e.,  $B \in {}^Q\mathcal{MA}^Q$ . Define  $R : A \otimes B \rightarrow B \otimes A$  and  $T : B \otimes A \rightarrow B \otimes A$  as  $R(a \otimes b) = \sum b_0 \otimes a \triangleleft b_{(1)}$  and  $T(b \otimes a) = \sum b_0 \otimes b_{(-1)} \triangleright a$ , respectively, for all  $a \in A$  and  $b \in B$ . Then, by Eq. (2.1.1), we have the generalized L-R-twisted tensor product  $B \diamond_r^Q A$  with the multiplication by

$$(b' \diamond a')(b \diamond a) = \sum b'_0 b_0 \diamond (a' \triangleleft b_{(1)}) (b'_{(-1)} \triangleright a) \quad (2.4.1)$$

for  $a, a' \in A$  and  $b, b' \in B$ .

Similarly, Define  $R : B \otimes A \rightarrow A \otimes B, b \otimes a \mapsto \sum (b_{(-1)} \triangleright a) \otimes b_0$  and  $T : A \otimes B \rightarrow A \otimes B, a \otimes b \mapsto \sum (a \triangleleft b_{(1)}) \otimes b_0$  for all  $a \in A$  and  $b \in B$ . Then, by Eq. (2.1.1), we can define  $A \diamond_l^Q B = A \otimes B$  as  $k$ -vector space with multiplication

$$(a \diamond b)(c \diamond d) = \sum (a \triangleleft d_{(1)}) (b_{(-1)} \triangleright c) \diamond b_0 d_0 \quad (2.4.2)$$

for  $a, c \in A, b, d \in B$ .

**Example 2.4.1.** (1) Let  $A$  be a right  $Q$ -module algebra. Then  $A$  becomes an  $Q$ -bimodule algebra, with left  $Q$ -action given via  $\varepsilon$ . In this case the multiplication of  $B \diamond_r^Q A$  becomes

$$(b' \diamond a')(b \diamond a) = \sum b'_0 b_0 \diamond (a' \triangleleft b_{(1)}) a$$

for  $a, a' \in A, b, b' \in B$ , hence in this case  $B \diamond_r^Q A$  coincides with the generalized smash product  $B \#_r^Q A$ .

(2) We note that  $Q$  itself is a  $Q$ -bicomodule algebra. So, in this case, the multiplication of  $Q \diamond_r^Q A$  specializes to

$$(x \diamond a)(y \diamond b) = \sum x_2 y_1 \diamond (a \triangleleft y_2) (x_1 \triangleright b)$$

for  $a, b \in A, x, y \in Q$ . If the left  $Q$ -action is trivial, then  $Q \diamond_r^Q A$  coincides with the smash product  $Q \#_r^Q A$

(3) If we consider a usual Hopf algebra  $Q$ , and assume that  $A$  is a  $Q$ -bimodule algebra. Define  $R : Q \otimes A \rightarrow A \otimes Q, q \otimes a \mapsto \sum q_1 \triangleright a \otimes q_2$  and  $T : A \otimes Q \rightarrow$

$A \otimes Q, a \otimes q \mapsto \sum a \triangleleft q_2 \otimes q_1$  for all  $a \in A$  and  $q \in Q$ . Then, by Eq. (2.1.1), we can define  $A \diamond_l^Q Q = A \otimes Q$  as  $k$ -vector space with multiplication

$$(a \diamond q)(b \diamond p) = \sum (a \triangleleft p_2)(q_1 \triangleright b) \diamond q_2 p_1 \quad (2.4.3)$$

for  $a, b \in A, q, p \in Q$ . If  $T(a \otimes q) = \sum a \triangleleft q_1 \otimes q_2$  and  $R(q \otimes a) = \sum q_2 \triangleright a \otimes q_1$ , then the conditions (2.11) and (2.12) hold. In fact, we have

$$\begin{aligned} & \sum (S(a_1)_T a_2)_R \# (S(q_1)_R q_2)_T \\ &= \sum ((S(a_1) \triangleleft (S(q_1)_R q_2)_2) a_2)_R \# (S(q_1)_R q_2)_1 \\ &= \sum ((S(a_1) \triangleleft (S(q_1)_{R2} q_3)) a_2)_R \# (S(q_1)_{R1} q_2) \\ &= \sum S(q_1)_1 \triangleright [(S(a_1) \triangleleft (S(q_1)_3 q_3)) a_2] \# (S(q_1)_2 q_2) \\ &= \sum [S(q_1)_1 \triangleright (S(a_1) \triangleleft (S(q_1)_4 q_3))] [S(q_1)_2 \triangleright a_2] \# (S(q_1)_3 q_2) \\ &= \sum [S(q_1)_2 \triangleright (S(a_1) \triangleleft (S(q_1)_4 q_3))] [S(q_1)_3 \triangleright a_2] \# (S(q_1)_1 q_2), \\ & \quad \text{by } R(q \otimes a) = \sum q_2 \triangleright a \otimes q_1 \\ &= \sum [S(q_3) \triangleright (S(a_1) \triangleleft (S(q_1) q_6))] [S(q_2) \triangleright a_2] \# (S(q_4) q_5) \\ &= \sum [S(q_3) \triangleright (S(a_1) \triangleleft (S(q_1) q_4))] [S(q_2) \triangleright a_2] \# 1_Q \\ &= \sum S(q_2) \triangleright [(S(a_1) \triangleleft (S(q_1) q_3)) a_2] \# 1_Q \\ &= \sum S(q_1) \triangleright [(S(a_1) \triangleleft (S(q_2) q_3)) a_2] \# 1_Q \\ & \quad \text{by } T(a \otimes q) = \sum a \triangleleft q_1 \otimes q_2 \\ &= \sum S(q) \triangleright (S(a_1) a_2) \# 1_Q \\ &= \varepsilon_{RT}(a \# q)(1_A \# 1_Q), \end{aligned}$$

and so Eq.(2.11) is obtained. Similarly, we have

$$\sum (a_1 S(a_2)_R)_T \# (b_1 S(b_2)_T)_R = \varepsilon_{RT}(a \# b)(1_A \# 1_B).$$

**Proposition 2.4.2.**  $B \diamond^Q A$  and  $A \diamond_l^Q B$  as above are associative algebras.

The proof of this result is straightforward.

**Proposition 2.4.3.** Let  $A$  be a  $Q$ -bimodule algebra and let  $B$  be a  $Q$ -bicomodule algebra. Then  $B \diamond_r^Q A \cong B \star_r^Q A$  and  $A \diamond_l^Q B \cong A \star_l^Q B$ .

**Proof.** Define  $\Phi : B \star_r^Q A \longrightarrow B \diamond_r^Q A$  as

$$b \star a \mapsto \sum b_0 \diamond (b_{(-1)} \triangleright a)$$

for all  $a \in A$  and  $b \in B$ . First, it is not hard to check that the map  $\Phi$  has the inverse  $\Psi : B \diamond_r^Q A \longrightarrow B \star_r^Q A$  given by

$$b \diamond a \mapsto \sum b_0 \star (S^{-1}(b_{(-1)}) \triangleright a)$$

Then we check that  $\Phi$  is a homomorphism as follows.

$$\begin{aligned}
\Phi((b' \star a')(b \star a)) &= \sum \Phi(b'b_0 \star (S^{-1}(b_{(-1)}) \triangleright a' \triangleleft b_{(1)})a) \\
&= \sum (b'b_0)_0 \diamond (b'b_0)_{(-1)} \triangleright [(S^{-1}(b_{(-1)}) \triangleright a' \triangleleft b_{(1)})a] \\
&= \sum b'_0 b_0 \diamond [(b'_{(-1)1} \triangleright a' \triangleleft b_{(1)})][(b'_{(-1)2} b_{(-1)} \triangleleft a)] \\
&= \sum (b'_0 \diamond (b'_{(-1)} \triangleright a'))(b_0 \diamond (b_{(-1)} \triangleright a)) = \Phi(b' \star a')\Phi(b \star a)
\end{aligned}$$

for  $a, a' \in A, b, b' \in B$ .

Similarly for  $A \diamond_l^Q B$ . This completes the proof. ■

**Example 2.4.4.** (1) Let  $Q$  be a regular multiplier Hopf algebra and let  $B$  be a  $Q$ -bicomodule algebra. If  $A$  is a left  $Q$ -module algebra regarded as an  $Q$ -bimodule algebra with trivial right  $Q$ -action, then  $A \star_l^Q B$  and  $A \diamond_l^Q B$  both coincide with  $A \#_l^Q B$ , and the isomorphism is just the identity.

(2) If  $B = Q$ , the maps  $\Phi : B \star_r^Q A \longrightarrow B \diamond_r^Q A$  and  $\Psi : B \diamond_r^Q A \longrightarrow B \star_r^Q A$  become:

$$\Phi(b \star a) = \sum b_1 \diamond (b_2 \triangleright a), \quad \text{and} \quad \Psi(b \diamond a) = \sum b_2 \star (S^{-1}(b_1) \triangleright a)$$

for all  $a \in A$  and  $b \in B$ .

Let now  $Q$  be a multiplier Hopf algebra,  $A$  a  $Q$ -bimodule algebra and  $B$  a  $Q$ -bicomodule algebra. Let  $C$  be an algebra in the Yetter-Drinfeld category  ${}^Q_Q\mathcal{YD}$  (see [De]) that is  $C$  is both a left  $Q$ -module algebra and a left  $Q$ -comodule algebra. These two structures are crossed via the Yetter-Drinfeld compatibility condition in the following sense. For all  $c \in C$  and  $x, y \in Q$  we require

$$\sum (x_1 \triangleright c)_{(-1)} x_2 y \otimes (x_1 \triangleright c)_0 = \sum x_1 c_{(-1)} y \otimes (x_2 \triangleright c_0). \quad (2.4.4)$$

Consider first the generalized smash product  $A \#_l^Q C$ , an associative algebra. From Eq.(2.4.4), it follows that  $A \#_l^Q C$  becomes a  $Q$ -bimodule algebra, with  $Q$ -actions

$$\begin{aligned}
x \triangleright (a \# c) &= \sum x_1 \triangleright a \# x_2 \triangleright c \\
(a \# c) \triangleleft x &= a \triangleleft x \otimes c
\end{aligned}$$

for all  $x \in Q, a \in A$  and  $c \in C$ , hence we may consider the algebra  $(A \#_l^Q C) \diamond_l^Q B$ . Similarly, for  $B \diamond_r^Q (A \#_l^Q C)$ .

Meanwhile, consider the generalized smash product  $C \#_l^Q B$ , an associative algebra. Using the condition Eq.(2.4.4), one can see that  $C \#_l^Q B$  becomes a  $Q$ -bicomodule algebra, with  $Q$ -coactions:

$$\begin{aligned}\Gamma : C \#_l^Q B &\longrightarrow Q \otimes (C \#_l^Q B), & \Gamma(a \# b) &= \sum a_{(-1)} b_{(-1)} \otimes (a_0 \# b_0) \\ \Upsilon : C \#_l^Q B &\longrightarrow (C \#_l^Q B) \otimes Q, & \Upsilon(a \# b) &= \sum (a \otimes b_0) \otimes b_{(1)}\end{aligned}$$

for all  $a \in A$  and  $b \in B$ , hence we may consider the algebra  $A \diamond_l^Q (C \#_l^Q B)$ .

**Proposition 2.4.5.** We have an algebra isomorphism  $(A \#_l^Q C) \diamond_l^Q B \cong A \diamond_l^Q (C \#_l^Q B)$ , given by the trivial identification.

**Proof.** We compute the multiplication in  $(A \#_l^Q C) \diamond_l^Q B$  as follows:

$$\begin{aligned}& ((a \# c) \diamond b) ((a' \# c') \diamond b') \\ &= \sum [((a \# c) \triangleleft b'_{(1)}) (b_{(-1)} \triangleright (a' \# c'))] \diamond b_0 b'_0 \\ &= \sum [((a \triangleleft b'_{(1)} \# c)) (b_{(-1)1} \triangleright a' \# b_{(-1)2} \triangleright c')] \diamond b_0 b'_0 \\ &= \sum (a \triangleleft b'_{(1)}) (c_{(-1)} b_{(-1)1} \triangleright a') \# c_0 (b_{(-1)2} \triangleright c') \diamond b_0 b'_0.\end{aligned}$$

The multiplication in  $A \diamond_l^Q (C \#_l^Q B)$  is:

$$\begin{aligned}& (a \diamond (c \# b)) (a' \diamond (c' \# b')) \\ &= \sum (a \triangleleft (c' \# b')_{(1)}) ((c \# b)_{(-1)} \triangleright a') \diamond (c \# b)_0 (c' \# b')_0 \\ &= \sum (a \triangleleft b'_{(1)}) (c_{(-1)} b_{(-1)} \triangleright a') \diamond (c_0 \# b_0) (c' \# b'_0) \\ &= \sum (a \triangleleft b'_{(1)}) (c_{(-1)} b_{(-1)} \triangleright a') \# c_0 (b_{0(-1)} \triangleright c') \diamond b_{00} b'_0.\end{aligned}$$

Hence the two multiplications coincide, completing the proof. ■

Since the L-R-smash product coincides with the generalized smash product if the right  $Q$ -action is trivial, we also obtain:

**Corollary 2.4.6.** If  $Q, A, B$  are as above and  $C$  is a left  $Q$ -module algebra, then we have an algebra isomorphism  $(C \#_l^Q A) \#_l^Q B \cong C \#_l^Q (A \#_l^Q B)$ , given by the trivial identification.

Let  $Q$  be a regular multiplier Hopf algebra. Recall that the Drinfel'd double  $D(Q)$  (generalizing the usual Drinfel'd double of a Hopf algebra) was introduced by Drabant and Van Daele in [Dr-VD] by a general procedure, and more explicit descriptions were obtained afterwards by Delvaux and Van Daele in [De-VD]. According to one of these descriptions, the algebra structure of  $D(Q)$  is just the twisted smash product  $\hat{Q} \star Q^{cop}$ . By transferring the whole structure of  $D(Q)$  via the map

$\Phi$ , we can thus obtain a new realization of  $D(Q)$ , having the L-R-smash product  $\widehat{Q} \diamond Q^{cop}$  for the algebra structure.

Let  $Q$  be a multiplier Hopf algebra. From [De-VD], an invertible element  $R \in M(Q \otimes Q)$  is called a Drinfeld twist (or a gauge transformation) if

$$(1 \otimes R)((\iota \otimes \Delta)(R)) = (R \otimes 1)((\Delta \otimes \iota)(R)) \quad (2.4.5)$$

$$(\varepsilon \otimes \iota)(R) = 1_Q = (\iota \otimes \varepsilon)(R). \quad (2.4.6)$$

If  $R = R^1 \otimes R^2 \in M(Q \otimes Q)$  is a Drinfeld twist with inverse  $R^{-1} = R^{-1} \otimes R^{-2} \in M(Q \otimes Q)$ , then we can define a new multiplier Hopf algebra  $Q_R$  (see, [Wa1]) with the same multiplication and counit as  $Q$ , for which the comultiplication and antipode are given by, for  $x \in Q$

$$\Delta_R(x) = R\Delta(x)R^{-1}, \quad (2.4.7)$$

$$S_R = T_R S(x) T_R^{-1} \quad \forall x \in Q \quad (2.4.8)$$

where  $T_R = R^1 S(R^2)$  is an invertible element of  $M(Q)$  with the inverse  $T_R^{-1} = S(R^{-1}) R^{-2}$ .

**Remark:** (1) It is easy to get that

$$((\iota \otimes \Delta)(R^{-1}))(1 \otimes R^{-1}) = ((\Delta \otimes \iota)(R^{-1}))(R^{-1} \otimes 1)$$

and

$$(\varepsilon \otimes \iota)(R^{-1}) = 1_Q = (\iota \otimes \varepsilon)(R^{-1}).$$

(2) Let  $x \in M(Q)$  be an invertible element such that  $\varepsilon(x) = 1_Q$ . If  $R$  is a twist for  $Q$  then so is  $R^x := R\Delta(x)(x^{-1} \otimes x^{-2})$ . The twists  $R$  and  $R^x$  are said to be gauge equivalent.

Let  $Q$  be a multiplier Hopf algebra,  $A$  a  $Q$ -bimodule algebra and  $R \in M(Q \otimes Q)$  a Drinfeld twist. If we introduce on  $A$  another multiplication, by  $a \circ b = (R^{-1} \triangleright a \triangleleft R^1)(R^{-2} \triangleright b \triangleleft R^2)$  for all  $a, b \in A$ , and denote this structure by  ${}_{R^{-1}}A_R$ , then  ${}_{R^{-1}}A_R$  is a  $Q_R$ -bimodule algebra, with the same  $Q$ -actions as  $A$ . We have

**Proposition 2.4.6.** Take the notations as above. Then  ${}_{R^{-1}}A_R$  is a  $Q_R$ -bimodule algebra.

**Proof.** We first want to show that the product in  ${}_{R^{-1}}A_R$  is associative and non-degenerate. Furthermore,  $1_Q \in A$  remains the unit in  $M({}_{R^{-1}}A_R)$ .

We compute

$$\begin{aligned} (a \circ b) \circ c &= [(R^{-1} \triangleright a \triangleleft R^1)(R^{-2} \triangleright b \triangleleft R^2)] \circ c \\ &= [(r_1^{-1} R^{-1} \triangleright a \triangleleft R^1 r_1^1)(r_2^{-1} R^{-2} \triangleright b \triangleleft R^2 r_2^1)](r^{-2} \triangleright c \triangleleft r^1) \end{aligned}$$

$$\begin{aligned}
&= (r^{-1} \triangleright a \triangleleft R^1)[(r_1^{-2} R^{-1} \triangleright b \triangleleft r^1 R_1^2)(r_2^{-2} R^{-2} \triangleright c \triangleleft r^1 R_2^2)] \\
&= (r^{-1} \triangleright a \triangleleft R^1)[r^{-2} \triangleright [(R^{-1} \triangleright b \triangleleft r^1)(R^{-2} \triangleright c \triangleleft r^1)] \triangleleft R^2] \\
&= a \circ [(R^{-1} \triangleright b \triangleleft r^1)(R^{-2} \triangleright c \triangleleft r^1)] \\
&= a \circ (b \circ c)
\end{aligned}$$

for all  $a, b, c \in A$

Then, one has to show that  ${}_{R^{-1}}A_R$  is a left  $Q_R$ -module algebra.

We note that  $\Delta_R(x) =: \sum x_{[1]} \otimes x_{[2]} = \sum R^1 x_1 R^{-1} \otimes R^2 x_2 R^{-2}$

$$\begin{aligned}
x \triangleright (a \circ b) &= x \triangleright [(R^{-1} \triangleright a \triangleleft R^1)(R^{-2} \triangleright b \triangleleft R^2)] \\
&= \sum [x_1 R^{-1} \triangleright a \triangleleft R^1][x_2 R^{-2} \triangleright b \triangleleft R^2] \\
&= \sum (r^{-1} R^1 x_1 R^{-1} \triangleright a \triangleleft r^1)(r^{-2} R^2 x_2 R^{-2} \triangleright b \triangleleft r^2) \\
&= \sum (R^1 x_1 R^{-1} \triangleright a) \circ (R^2 x_2 R^{-2} \triangleright b) \\
&= \sum (x_{[1]} \triangleright a) \circ (x_{[2]} \triangleright b)
\end{aligned}$$

for all  $x \in Q, a, b \in A$

Similarly, one can check that  ${}_{R^{-1}}A_R$  is a right  $Q_R$ -module algebra and that  ${}_{R^{-1}}A_R$  is a  $Q_R$ -bimodule. This finishes the proof. ■

Suppose that we have a left  $Q$ -comodule algebra  $B$ ; then on the algebra structure of  $B$  one can introduce a left  $Q_R$ -comodule algebra structure (denoted by  ${}^{R^{-1}}B$  in what follows) putting the same  $Q$ -coaction  $\Gamma$  as  $B$ . Similarly, if  $C$  is a right  $Q$ -comodule algebra, one can introduce on the algebra structure of  $C$  a right  $Q_R$ -comodule algebra structure (denoted by  $C^R$  in what follows) putting the same  $Q$ -coaction  $\Gamma$  as  $C$ . One may check that if  $B$  is a  $Q$ -bicomodule algebra, the left and right  $Q_R$ -comodule algebras  ${}^{R^{-1}}B$  respectively  $B^R$  actually define the structure of a  $Q_R$ -bicomodule algebra on  $B$ , denoted by  ${}^{R^{-1}}B^R$ .

Then the proof of the following result is straightforward.

**Proposition 2.4.7.** Take the notations as above. If  $B$  is a  $Q$ -bicomodule algebra, then  ${}^{R^{-1}}B^R$  is a  $Q_R$ -bicomodule algebra.

**Proposition 2.4.8.** Let  $A$  be a  $Q$ -bimodule algebra and  $B$  a  $Q$ -bicomodule algebra. Then we have two algebra isomorphisms:

$$B \diamond_r^Q A \cong {}^{R^{-1}}B^R \diamond_r^Q {}_{R^{-1}}A_R, \quad \text{and} \quad A \diamond_l^Q B \cong {}_{R^{-1}}A_R \diamond_l^Q {}^{Q^{R^{-1}}}B^R$$

given by the trivial identification.

**Proof.** We only compute the multiplication in  ${}^{R^{-1}}B^R \diamond_r^Q {}_{R^{-1}}A_R$ . Similar to  ${}_{R^{-1}}A_R \diamond_l^Q {}^{Q^{R^{-1}}}B^R$ .

$$(b' \diamond a')(b \diamond a)$$



$$\begin{aligned}
&= \sum b'_0 b_0 \diamond (a' \triangleleft b_{(1)}) \circ (b'_{(-1)} \triangleright a) \\
&= \sum b'_0 b_0 \diamond (R^{-1} \triangleright a' \triangleleft b_{(1)} R^1) (R^{-2} b'_{(-1)} \triangleright a \triangleleft R^2)
\end{aligned}$$

for  $a, a' \in A, b, b' \in B$ .

which is the multiplication of

This finishes the proof. ■

By Proposition 2.4.3, we have

**Corollary 2.4.9.** Take the notations and assumptions as above. We have

$$B \star_r^Q A \cong {}^{R^{-1}} B^R \diamond_r^Q {}_{R^{-1}} A_R, \quad \text{and} \quad A \star_l^Q B \cong {}_{R^{-1}} A_R \diamond_l^Q {}^{Q R^{-1}} B^R.$$

### 3. Two-times twisted tensor products

In this section, we will study the notion of a two-times twisted tensor product.

#### 3.1 Two-sided twisted tensor products

Let  $A, B, C$  be algebras and let  $R : C \otimes A \longrightarrow A \otimes C$  and  $T : B \otimes C \longrightarrow C \otimes B$  be  $k$ -linear maps. Then we will write  $R(c \otimes a) = a_R \otimes c_R = a_r \otimes c_r$  and  $T(b \otimes c) = c_T \otimes b_T = c_t \otimes b_t$  for all  $a \in A, b \in B$  and  $c \in C$ . One defines two-sided twisted tensor product  $A \#_R C \#_T B = A \otimes C \otimes B$  as spaces with a new multiplication defined by the formula:

$$m_{A \#_R C \#_T B} = (m_A \otimes m_C \otimes m_B) \circ (\iota_A \otimes R \otimes T \otimes \iota_B) \circ (\iota_{A \otimes C} \otimes \sigma \otimes \iota_{C \otimes B})$$

or

$$(a \boxplus c \boxplus b)(a' \boxplus c' \boxplus b') = a a'_R \otimes c_R c'_T \otimes b_T b'$$

for all  $a, a' \in A, b, b' \in B$  and  $c, c' \in C$ , where  $\sigma$  denotes the usual flip map on  $B \otimes A$ .

Let  $Q, L$  be multiplier Hopf algebras. Let  $A$  be in  ${}^Q \mathcal{MA}$ ,  $B \in \mathcal{MA}_L$ , and let  $C$  be in  ${}^Q \mathcal{MA}^L$ . If we define  $R : C \otimes A \longrightarrow A \otimes C$  and  $T : B \otimes C \longrightarrow C \otimes B$ , respectively by  $R(c \otimes a) = \sum c_{(-1)} \triangleright a \otimes c_0$  and  $T(b \otimes c) = c_0 \otimes b \triangleleft c_{(1)}$  for all  $a \in A, b \in B$  and  $c \in C$ , then we have a two-sided twisted tensor product  $A \boxplus_l^Q C \boxplus_r^L B$  with the multiplication given by

$$(a \boxplus c \boxplus b)(a' \boxplus c' \boxplus b') = \sum a(c_{(-1)} \triangleright a') \boxplus c_0 c'_0 \boxplus (b \triangleleft c'_{(1)}) b' \quad (3.1.1)$$

for  $a, a' \in A, b, b' \in B, c, c' \in C$ , here we write  $a \boxplus c \boxplus b$  for  $a \otimes c \otimes b$ .

**Proposition 3.1.1.**  $A \boxminus_l^Q C \boxminus_r^L B$  defined above is an associative algebra.

Note that, given  $A, B$  as above,  $A \otimes B$  becomes a  $Q$ -bimodule algebra, with  $Q$ -actions

$$x \triangleright (a \otimes b) \triangleleft y = x \triangleright a \otimes b \triangleleft y$$

for all  $a \in A, x, y \in Q, b \in B$ .

Then we have

**Proposition 3.1.2.** Let  $A, B, C, Q$  be as above. Then we have the following algebra isomorphisms

- (1)  $\Phi : (A \otimes B) \diamond_l^L C \cong A \boxminus_l^Q C \boxminus_r^L B, \quad \Phi((a \otimes b) \diamond c) = a \boxminus c \boxminus b;$
- (2)  $\Psi : C \diamond_r^Q (A \otimes B) \cong A \boxminus_l^Q C \boxminus_r^Q B, \quad \Psi(c \diamond (a \otimes b)) = a \boxminus c \boxminus b;$
- (3)  $(A \otimes B) \star_l^L C \cong (A \otimes B) \diamond_l^L C \cong C \diamond_r^Q (A \otimes B) \cong C \star_r^Q (A \otimes B);$
- (4)  $(A \#_l^Q C) \#_r^L B = A \#_l^Q (C \#_r^L B) = A \boxminus_l^Q C \boxminus_r^L B.$

**Proof.** (1) We do calculations as follows:

$$\begin{aligned}
& \Phi[((a \otimes b) \diamond c)((a' \otimes b') \diamond c')] \\
&= \sum \Phi[((a \otimes b) \triangleleft c'_{(1)})(c_{(-1)} \triangleright (a' \otimes b')) \diamond c_0 c'_0], \quad \text{by Eq.(2.4.3)} \\
&= \sum \Phi[(a \otimes b \triangleleft c'_{(1)})(c_{(-1)} \triangleright a' \otimes b') \diamond c_0 c'_0] \\
&= \sum \Phi[a(c_{(-1)} \triangleright a') \otimes (b \triangleleft c'_{(1)}) b' \diamond c_0 c'_0] \\
&= \sum a(c_{(-1)} \triangleright a') \boxminus c_0 c'_0 \boxminus (b \triangleleft c'_{(1)}) b' \\
&= (a \# c \# b)(a' \boxminus c' \boxminus b') \\
&= \Phi[((a \otimes b) \diamond c)] \Phi[((a' \otimes b') \diamond c')]
\end{aligned}$$

for  $a, a' \in A, b, b' \in B, c, c' \in C$ .

(2) Similar to (1).

(3) follows (1), (2) and Proposition 2.4.3.

(4) We note that  $A \#_l^Q C$  has an induced right  $Q$ -comodule coaction by the right coaction of  $Q$  on  $C$ , i.e.,  $\Upsilon : A \#_l^Q C \longrightarrow A \#_l^Q C \otimes Q, \Upsilon(a \# c) = a \# c_0 \otimes c_{(1)}$  for all  $a \in A, c \in C$ . Hence,  $(A \#_l^Q C) \#_r^Q B = A \boxminus_l^Q C \boxminus_r^Q B$ . Similarly,  $A \#_l^Q (C \#_r^Q B) = A \boxminus_l^Q C \boxminus_r^Q B$ .

This finishes the proof. ■

### 3.2 Two-sided $L$ - $R$ -smash products

Let  $Q$  be a multiplier Hopf algebra,  $A$  a right  $Q$ -comodule algebra,  $B$  a left  $Q$ -comodule algebra and  $C$  a  $Q$ -bimodule algebra. Define on  $A \otimes C \otimes B$  a multiplication by the formula

$$(a \boxtimes c \boxtimes b)(a' \boxtimes c' \boxtimes b') = \sum aa'_0 \boxtimes (c \triangleleft a'_{(1)})(b_{(-1)} \triangleright c') \boxtimes b_0 b' \quad (3.2.1)$$

for  $a, a' \in A, b, b' \in B, c, c' \in C$ .

Then this multiplication yields an associative algebra structure, denoted by  $A \boxtimes_r^Q C \boxtimes_l^Q B$ .

Note that, given  $A, B$  as above,  $A \otimes B$  becomes an  $Q$ -bicomodule algebra, with the following structure:

$$\Gamma : A \otimes B \longrightarrow M(Q \otimes (A \otimes B)), a \otimes b \mapsto b_{(-1)} \otimes (a \otimes b_0) =: (a \otimes b)_{[-1]} \otimes (a \otimes b)_0,$$

and

$$\Upsilon : A \otimes B \longrightarrow M((A \otimes B) \otimes Q), a \otimes b \mapsto (a_0 \otimes b) \otimes a_{(1)} =: (a \otimes b)_0 \otimes (a \otimes b)_{[1]}.$$

**Proposition 3.2.1.** If  $Q, A, B, C$  are as above, then we have the following algebra isomorphisms:

- (1)  $\phi : (A \otimes B) \diamond_r^Q C \cong A \boxtimes_r^Q C \boxtimes_l^Q B, \quad \phi((a \otimes b) \diamond c) = a \boxtimes c \boxtimes b;$
- (2)  $\psi : C \diamond_l^Q (A \otimes B) \cong A \boxtimes_r^Q C \boxtimes_l^Q B, \quad \psi(c \diamond (a \otimes b)) = a \boxtimes c \boxtimes b;$
- (3)  $(A \otimes B) \star_r^Q C \cong (A \otimes B) \diamond_r^Q C \cong C \diamond_l^Q (A \otimes B) \cong C \star_l^Q (A \otimes B);$
- (4)  $(A \#_r^Q C) \#_l^Q B = A \#_r^Q (C \#_l^Q B) = A \boxtimes_r^Q C \boxtimes_l^Q B.$

**Proof.** (1) We compute:

$$\begin{aligned} & \phi[((a' \otimes b') \diamond c')((a \otimes b) \diamond c)] \\ &= \sum \phi[(a' \otimes b')_0(a \otimes b)_0 \diamond (c' \triangleleft (a \otimes b)_{[1]})((a' \otimes b')_{[-1]} \triangleright c)] \\ &= \sum \phi[(a' \otimes b'_0)(a_0 \otimes b) \diamond (c' \triangleleft a_{(1)})(b'_{(-1)} \triangleright c)] \\ &= \sum \phi[(a' a_0 \otimes b'_0 b) \diamond (c' \triangleleft a_{(1)})(b'_{(-1)} \triangleright c)] \\ &= \sum a' a_0 \boxtimes (c' \triangleleft a_{(1)})(b'_{(-1)} \triangleright c) \boxtimes b'_0 b \\ &= \phi((a' \otimes b') \diamond c') \phi((a \otimes b) \diamond c). \end{aligned}$$

(2) and (3) Obvious.

(4) We note that  $A \#_r^Q C$  has an induced  $Q$ -bimodule algebra by the one on  $C$ . This finishes the proof. ■

## 4. Twisted products

In this section we will introduce the notion of a Long module algebra and study the the twisted product of the given algebra.

**Definition 4.1.** Let  $Q$  be a multiplier Hopf algebra.

(1) Let  $A$  be a left  $Q$ -module algebra and a left  $Q$ -comodule algebra.  $A$  is called a *left-left  $Q$ -Long module algebra* if the following condition holds:

$$\Gamma(x \triangleright a)(y \otimes 1) = \sum a_{(-1)} y \otimes x \triangleright a_0 \quad (4.1)$$

for all  $x, y \in Q, a \in A$ . The category of all left  $Q$ -Long module algebras is denoted by  ${}^Q\mathcal{LA}$ .

(2) Let  $A$  be a left  $Q$ -module algebra and a right  $Q$ -comodule algebra.  $A$  is called a *left-right  $Q$ -Long module algebra* if the following condition holds:

$$\Upsilon(x \triangleright a)(1 \otimes y) = \sum x \triangleright a_0 \otimes a_{(1)}y \quad (4. 2)$$

for all  $x, y \in Q, a \in A$ . The category of all left-right  $Q$ -Long module algebras is denoted by  ${}^Q\mathcal{LA}^Q$ .

(3) Let  $A$  be a right  $Q$ -module algebra and a right  $Q$ -comodule algebra.  $A$  is called a *right-right  $Q$ -Long module algebra* if the following condition holds:

$$\Upsilon(a \triangleleft x)(1 \otimes y) = \sum a_0 \triangleleft x \otimes a_{(1)}y \quad (4. 3)$$

for all  $x, y \in Q, a \in A$ . The category of all right-right  $Q$ -Long module algebras is denoted by  $\mathcal{LA}_Q^Q$ .

(4) Let  $A$  be a right  $Q$ -module algebra and a left  $Q$ -comodule algebra.  $A$  is called a *right-left  $Q$ -Long module algebra* if the following condition holds:

$$\Gamma(a \triangleleft x)(y \otimes 1) = \sum a_{(-1)}y \otimes a_0 \triangleleft x \quad (4. 4)$$

for all  $x, y \in Q, a \in A$ . The category of all right-left  $Q$ -Long module algebras is denoted by  ${}^Q\mathcal{LA}_Q$ .

Let  $A \in {}^Q\mathcal{LA}$ . If we define a new multiplication on  $A$ , by

$$a \bullet b = \sum a_0(a_{(-1)} \triangleright b) \quad \forall a, b \in A \quad (4. 5)$$

then this multiplication defines a new algebra structure on  $A$ . The product  $\bullet$  is called the left twisted product.

**Example 4.2.** Let  $Q$  be a regular multiplier Hopf algebra with bijective antipode  $S$ ,  $A$  a  $Q$ -bimodule algebra and  $B$  a  $Q$ -bicomodule algebra. Then  $A$  becomes a left  $Q \otimes Q^{op}$ -module algebra with the following structure:

$$(x \otimes y) \triangleright a = x \triangleright a \triangleleft y,$$

and  $B$  becomes a left  $Q \otimes Q^{op}$ -comodule algebra with the following structure

$$\Gamma : B \longrightarrow (Q \otimes Q^{op}) \otimes B, \quad b \mapsto \sum b_{(-1)} \otimes S^{-1}(b_{(1)}) \otimes b_0.$$

It is easy to check that  $A \otimes B$  is a left  $Q \otimes Q^{op}$ -Long module algebra with the natural structure. The corresponding twisted product on  $A \otimes B$  is

$$\begin{aligned} (a \otimes b) \bullet (a' \otimes b') &= (a \otimes b)_0[(a \otimes b)_{(-1)} \triangleright (a' \otimes b')] \\ &= (a \otimes b_0)[b_{(-1)} \triangleright a' \triangleleft S^{-1}(b_{(1)}) \otimes b'] \\ &= a(b_{(-1)} \triangleright a' \triangleleft S^{-1}(b_{(1)})) \otimes b_0b' \end{aligned}$$

for all  $a, a' \in A$  and  $b, b' \in B$ , and this is exactly the multiplication of the generalized right twisted smash product.

Similarly, for  $A \in \mathcal{LA}_Q^Q$ . If we define a new multiplication on  $A$ , by

$$a * b = \sum (a \triangleleft b_{(1)}) b_0 \quad \forall a, b \in A \quad (4.6)$$

then this multiplication defines a new algebra structure on  $A$ . The product  $*$  is called the right twisted product.

Let  $Q$  be a regular multiplier Hopf algebra with bijective antipode  $S$ , let  $A$  be a  $Q$ -bimodule algebra and a  $Q$ -bicomodule algebra such that  $A \in {}^Q_Q\mathcal{LA}_Q^Q$ , called *four sides Long module category*. Then we define a new multiplication on  $A$  by

$$a \otimes b = \sum (a_0 \triangleleft b_{(1)})(a_{(-1)} \triangleright b_0) \quad \forall a, b \in A \quad (4.7)$$

and call it the L-R-twisted product.

Then it is easy to check that  $A$  is a left  $Q \otimes Q^{op}$ -Long module algebra with the natural structure (see Example 4.2). The corresponding twisted product on  $A$  is

$$a \bullet b = \sum a_0(a_{(-1)} \triangleright b \triangleleft S^{-1}(a_{(1)})) \quad \forall a, b \in A. \quad (4.8)$$

**Proposition 4.3.**  $(A, \otimes)$  is an associative algebra.

**Proof.** We do calculations as follows:

$$\begin{aligned} & (a \otimes b) \otimes c \\ &= \sum [(a_0 \triangleleft b_{(1)})(a_{(-1)} \triangleright b_0)] \otimes c \\ &= \sum [(a_0 \triangleleft b_{(1)})(a_{(-1)} \triangleright b_0)]_0 \triangleleft c_{(1)} ([ (a_0 \triangleleft b_{(1)})(a_{(-1)} \triangleright b_0) ]_{(-1)} \triangleright c_0) \\ &= \sum [(a_0 \triangleleft b_{(1)})_0 (a_{(-1)} \triangleright b_0)_0] \triangleleft c_{(1)} ([ (a_0 \triangleleft b_{(1)})_{(-1)} (a_{(-1)} \triangleright b_0)_{(-1)} ] \triangleright c_0) \\ &= \sum [(a_0 \triangleleft b_{(1)})(a_{(-1)} \triangleright b_0)] \triangleleft c_{(1)} ((a_{0(-1)} b_{0(-1)}) \triangleright c_0) \\ &= \sum (a_0 \triangleleft [b_{0(1)} c_{0(1)}]) (a_{(-1)} \triangleright [(b_0 \triangleleft c_{(1)})(b_{(-1)} \triangleright c_0)]) \\ &= \sum (a_0 \triangleleft [(b_0 \triangleleft c_{(1)})_{(1)} (b_{(-1)} \triangleright c_0)_{(1)}]) (a_{(-1)} \triangleright [(b_0 \triangleleft c_{(1)})_0 (b_{(-1)} \triangleright c_0)_0]) \\ &= \sum (a_0 \triangleleft [(b_0 \triangleleft c_{(1)})(b_{(-1)} \triangleright c_0)]_{(1)}) (a_{(-1)} \triangleright [(b_0 \triangleleft c_{(1)})(b_{(-1)} \triangleright c_0)]_0) \\ &= \sum a \otimes [(b_0 \triangleleft c_{(1)})(b_{(-1)} \triangleright c_0)] \\ &= \sum a \otimes [(b_0 \triangleleft c_{(1)})(b_{(-1)} \triangleright c_0)] \\ &= a \otimes (b \otimes c) \end{aligned}$$

for all  $a, b, c \in A$ .

This finishes the proof. ■

Four sides Long module algebra category is in particular regarded as a left-left Long module algebra category, but in general the corresponding twisted products  $\circledast$  and respectively  $\bullet$  are different. On the other hand, any a left-left Long module algebra category can be regarded as a four sides Long module algebra category with trivial right action and coaction, and in this case the corresponding twisted products coincide.

**Proposition 4.4.** Let  $A \in {}^Q\mathcal{LA}_Q^Q$ . With notation as before. Then the L-R-twisted product  $\circledast$  can be obtained as a left twisting followed by a right twisting and also vice versa.

**Proof.** First consider the left twisted product algebra  $(A, \bullet)$ ; it is easy to see that  $(A, \bullet) \in \mathcal{LA}_Q^Q$ , and the corresponding right twisted product becomes:

$$\begin{aligned} a * b &= \sum (a \triangleleft b_{(1)}) \bullet b_0 \\ &= \sum (a \triangleleft b_{(1)})_0 ((a \triangleleft b_{(1)})_{(-1)} \triangleright b_0) \\ &= \sum (a_0 \triangleleft b_{(1)}) (a_{(-1)} \triangleright b_0) \\ &= a \circledast b \end{aligned}$$

for all  $a, b \in A$ .

Similarly, one can start with the right twisted product algebra  $(A, *)$ , for which  $(A, *) \in {}^Q\mathcal{LA}$ , and the corresponding left twisted product coincides with the L-R-twisted product.

This finishes the proof. ■

**Example 4.5.** Let  $Q$  be a regular multiplier Hopf algebra,  $A$  a  $Q$ -bimodule algebra and  $B$  a  $Q$ -bicomodule algebra. Take the algebra  $A \otimes B$ , which becomes a  $Q$ -bimodule algebra with actions  $x \triangleright (a \otimes b) = x \triangleright a \otimes b$  and  $(a \otimes b) \triangleleft x = a \triangleleft x \otimes b$ , for all  $x \in Q, a \in A, b \in B$ , and a  $Q$ -bicomodule algebra, with coactions  $\Gamma : A \otimes B \rightarrow Q \otimes (A \otimes B)$ ,  $a \otimes b \mapsto \sum b_{(-1)} \otimes (a \otimes b_0)$  and  $\Upsilon : A \otimes B \rightarrow (A \otimes B) \otimes Q$ ,  $a \otimes b \mapsto \sum (a \otimes b_0) \otimes b_{(1)}$ . Moreover, one checks that  $A \otimes B \in {}^Q\mathcal{LA}_Q^Q$ , hence we have an L-R-twisting datum for  $A \otimes B$ . The corresponding L-R-twisted product is:

$$\begin{aligned} (a \otimes b) \circledast (a' \otimes b') &= \sum ((a \otimes b)_0 \triangleleft (a' \otimes b')_{(1)}) ((a \otimes b)_{(-1)} \triangleright (a' \otimes b')_0) \\ &= \sum ((a \otimes b_0) \triangleleft b'_{(1)}) (b_{(-1)} \triangleright (a' \otimes b'_0)) \\ &= \sum (a \triangleleft b'_{(1)} \otimes b_0) (b_{(-1)} \triangleright a' \otimes b'_0) \\ &= \sum (a \triangleleft b'_{(1)}) (b_{(-1)} \triangleright a') \otimes b_0 b'_0 \end{aligned}$$

for all  $a, a' \in A$  and  $b, b' \in B$ , and this is exactly the multiplication of the generalized right twisted smash product. and this is exactly the multiplication of the L-R-smash product  $A \diamond_l^Q B$ .

**Theorem 4.6.** With notation as above, let  $Q$  be a regular multiplier Hopf algebra with bijective antipode  $S$ . Let  $A \in {}^Q_Q \mathcal{LA}_Q^Q$ . Then  $A$  is a left  $Q \otimes Q^{op}$ -Long module algebra (see Example 4.2). Moreover, the corresponding twisted algebras  $(A, \bullet)$  and  $(A, \otimes)$  are isomorphic, and the isomorphism is defined by:

$$\alpha : (A, \bullet) \longrightarrow (A, \otimes), \quad a \mapsto \sum a_0 \triangleleft a_{(1)}; \quad (4.9)$$

$$\alpha^{-1} : (A, \otimes) \longrightarrow (A, \bullet), \quad a \mapsto \sum a_0 \triangleleft S^{-1}(a_{(1)}) \quad (4.10)$$

In particular, we obtain  $A \diamond_r^Q A \cong A \star_r^Q A$  and  $A \diamond_l^Q A \cong A \star_l^Q A$ .

**Proof.** We only prove that  $\alpha$  is an algebra isomorphism. It is easy to check that  $\alpha \alpha^{-1} = \alpha^{-1} \alpha = id$ . Hence we only have to check that  $\alpha$  is multiplicative:

$$\begin{aligned} \alpha(a \bullet b) &= \sum \alpha(a_0(a_{(-1)} \triangleright b \triangleleft S^{-1}(a_{(1)}))) \quad \text{by Eq.(4.8)} \\ &= \sum (a_0(a_{(-1)} \triangleright b \triangleleft S^{-1}(a_{(1)})))_0 \triangleleft (a_0(a_{(-1)} \triangleright b \triangleleft S^{-1}(a_{(1)})))_{(1)} \\ &= \sum (a_0(a_{(-1)} \triangleright b \triangleleft S^{-1}(a_{(1)})))_0 \triangleleft (a_{0(1)}(a_{(-1)} \triangleright b)_{(1)}) \\ &= \sum (a_0(a_{(-1)} \triangleright b_0 \triangleleft S^{-1}(a_{(1)}))) \triangleleft (a_{(1)}b_{(1)}) \end{aligned}$$

and

$$\begin{aligned} \alpha(a) \otimes \alpha(b) &= \sum (a_0 \triangleleft a_{(1)}) \otimes (b_0 \triangleleft b_{(1)}) \\ &= \sum (a_0 \triangleleft a_{(1)}) \otimes (b_0 \triangleleft b_{(1)}) \\ &= \sum ((a_0 \triangleleft a_{(1)})_0 \triangleleft (b_0 \triangleleft b_{(1)})_{(1)})((a_0 \triangleleft a_{(1)})_{(-1)} \triangleright (b_0 \triangleleft b_{(1)})_0) \\ &= \sum ((a_0 \triangleleft a_{(1)}) \triangleleft (b_{0(1)}))(a_{0(-1)} \triangleright (b_0 \triangleleft b_{(1)})) \\ &= \sum ((a_0 \triangleleft a_{(1)}) \triangleleft (b_{(1)1}))(a_{(-1)} \triangleright b_0 \triangleleft b_{(1)2}) \\ &= \sum ((a_0 \triangleleft a_{(1)})(a_{(-1)} \triangleright b_0)) \triangleleft b_{(1)} \end{aligned}$$

and the proof is finished. ■

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